THE GEOMETRY OF ENTANGLEMENT: FROM PROJECTIVE DUALITY TO QUANTUM COMPUTER

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Abstract. Theses notes correspond to a series of three lectures given at the Institut Henri Poincaré between October 5 and October 9, 2024, as part of the program on Random Tensors hosted at the institute in October 2024. This course is primarily intended for Master and PhD students.

Introduction

The goal of these lectures is to present research I have conducted over the past 12 years on understanding the entanglement of multipartite pure quantum states from a geometric perspective. When I began working on this topic, my initial approach, inspired by the pioneering works of Heydari [21] and Miyake [36], was to use classical concepts from algebraic geometry to describe different classes of entanglement as distinct strata defined by algebraic varieties. These varieties can be characterized by the vanishing of invariants and covariants, following the principles of classical invariant theory, which I learned from my collaboration with Jean-Gabriel Luque and Jean-Yves Thibon [25–27].

With Peter Levay [24,33], and later with Luke Oeding [28,29], I employed representation theory techniques to provide invariants for a broader range of quantum systems, extending beyond multipartite qudits. More recently, with my students Hamza Jaffali [23,30,31], Henri de Boutray [9], and Grâce Amouzou [2], we have begun exploring how these geometric concepts can be applied to the study of quantum algorithms and how these 150-year-old ideas can be integrated into the ongoing development of quantum computing.

These notes, divided into three lectures, explain, through examples, this journey from classical algebraic geometry and invariants theory to experiments with quantum computers.

Setting and Notations. In these lectures, we will consider the following general setting:

- *Hilbert Space*: The Hilbert space is defined as the tensor product of n complex vector spaces:
$$\mathcal{H}_{d_1,\dots,d_n} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n},$$
where d_i is the dimension of the i-th vector space.
- *Pure n-Partite States*: A pure n-partite state is a normalized vector in the Hilbert space $\mathcal{H}_{d_1,\dots,d_n}$ with $n \geq 2$. Using Dirac notation $|0\rangle, \dots, |d_i - 1\rangle$ for the standard basis of $\mathbb{C}^{d_i}$, we express the state as:
$$|\psi\rangle = \sum_{(i_1,i_2,\dots,i_n)\in I} a_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle,$$
where $\sum_{(i_1,i_2,\dots,i_n)\in I} |a_{i_1 i_2 \dots i_n}|^2 = 1$ and
$$I = [\![0, d_1 - 1]\!] \times [\![0, d_2 - 1]\!] \times \cdots \times [\![0, d_n - 1]\!].$$

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Here, $|i_1 \dots i_n\rangle = |i_1\rangle \otimes \cdots \otimes |i_n\rangle$. Recall that in quantum mechanics, $|a_{i_1 \dots i_n}|^2$ corresponds the probability that $|\psi\rangle$ will be projected to $|i_1 \dots i_n\rangle$ when measured in the standard basis.

- *Separable and Entangled States*: A state $|\psi\rangle \in \mathcal{H}_{d_1,\dots,d_n}$ is called *separable* if there exist vectors $|\varphi_i\rangle \in \mathbb{C}^{d_i}$ for $i = 1, \dots, n$ such that:

$$|\psi\rangle = |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes \cdots \otimes |\varphi_n\rangle .$$

If no such decomposition exists, $|\psi\rangle$ is said to be *entangled*.

- *Groups of Local Actions*:
  - *Local Unitary Operations (LU)*: The group of local unitary transformations is given by:

$$\mathrm{LU} = U_{d_1}(\mathbb{C}) \times \cdots \times U_{d_n}(\mathbb{C}),$$

  where $U_{d_i}(\mathbb{C})$ is the unitary group on $\mathbb{C}^{d_i}$.
  - *Stochastic Local Operations with Classical Communication (SLOCC)*: The group of SLOCC transformations is:

$$\mathrm{SLOCC} = SL_{d_1}(\mathbb{C}) \times \cdots \times SL_{d_n}(\mathbb{C}),$$

  where $SL_{d_i}(\mathbb{C})$ is the special linear group of degree d_i over $\mathbb{C}$ [11].

In quantum information theory, entanglement is considered a resource for performing quantum protocols and plays a crucial role in quantum computation. As a resource, entanglement should be measured and classified. Since entanglement is a non-local property, its classification is performed up to local actions.

Given two quantum states $|\psi\rangle$ and $|\psi'\rangle$, the following natural questions arise:

- Are $|\psi\rangle$ and $|\psi'\rangle$ entangled? To what extent? Which one is *more* entangled?
- Are $|\psi\rangle$ and $|\psi'\rangle$ equivalent under local operations?
- Can we provide a classification of entanglement types in $\mathcal{H}_{d_1,\dots,d_n}$? To which entanglement classes do $|\psi\rangle$ and $|\psi'\rangle$ belong?

Remark 1. The concept of separability for pure quantum states aligns with the notion of a rank-one tensor. The rank of a tensor generalizes the rank of a matrix. A tensor $|\psi\rangle$ is said to be of *rank* r if and only if there exist r rank-one tensors (i.e., separable states) $|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_r\rangle$ such that:

$$|\psi\rangle = |\psi_1\rangle + |\psi_2\rangle + \cdots + |\psi_r\rangle ,$$

with r being minimal for this property. Notably, if $r \geq 2$, then $|\psi\rangle$ is entangled. The study of rank decompositions of tensors has gained significant interest over the past 20 years due to its rich geometric structure and wide range of practical applications, including computer vision, matrix multiplication algorithms, and quantum information theory (see [32]). Since the rank of a quantum state is invariant under SLOCC transformations, it can serve as an algebraic measure of entanglement [7].

Now, let's recall the basic framework of our geometric construction:

- *Projective Space*: Let V be a complex vector space of dimension $d+1$. The projective space $\mathbb{P}(V)$ is the space of lines through the origin in V, defined as:

$$\mathbb{P}(V) = \{[v] \mid v \in V \setminus \{0\}, [v] = [w]\ w = \lambda v \text{ for some } \lambda \in \mathbb{C}^*\}.$$

- *Projective Coordinates*: Consider a basis $(e_0, e_1, \dots, e_d)$ of V such that $v = (a_0, a_1, \dots, a_d)$ are the coordinates of $v \in V$. Then, the projective coordinates of the point $[v] \in \mathbb{P}(V)$ are given by:
$$[v] = [a_0 : a_1 : \cdots : a_d],$$
which are well-defined up to a non-zero scalar multiplication. The projective space $\mathbb{P}(V)$ is d-dimensional.
- *Algebraic Varieties*: An algebraic variety $X \subset \mathbb{P}(V)$ is defined as the zero locus of a collection of homogeneous polynomials. For a subset $Y \subset \mathbb{P}(V)$, the *Zariski closure* $\overline{Y} \subset \mathbb{P}(V)$ is the smallest algebraic variety containing Y.

Most of the examples of Hilbert spaces and quantum states discussed in this course will involve $2 \leq n \leq 5$ and $2 \leq d_i \leq 3$. Specifically, we will consider:

- **Toy Example**:
  - Two qubits: $n = 2$, $d_1 = d_2 = 2$.
  - Two qutrits: $n = 2$, $d_1 = d_2 = 3$.
- **Non-Trivial Examples (Fully Understood)**:
  - Three qubits: $n = 3$, $d_1 = d_2 = d_3 = 2$.
  - Tripartite systems with one qubit: $n = 3$, $d_1 = 2$, $2 \leq d_2, d_3 \leq 3$.
- **Advanced Examples (Expert Level)**:
  - Four qubits: $n = 4$, $d_i = 2$.
  - Three qutrits: $n = 3$, $d_i = 3$.
- **Open Problems**:
  - Systems with $n \geq 5$ or $n \geq 4$ and at least one qutrit.

1. The Geometry of Entangled States

In this first lecture, I introduce the concept of Segre varieties and the auxiliary varieties that can naturally be derived from them to stratify entanglement. I conclude with a simple algorithm for identifying the entanglement class of a pure three-qubit quantum state.

1.1. Stratification of the Projectivized Hilbert Space by Auxiliary Varieties.

In quantum mechanics, two quantum states $|\psi\rangle$ and $|\psi'\rangle$ such that $|\psi'\rangle = \lambda |\psi\rangle$ for some nonzero scalar $\lambda \in \mathbb{C}$ are physically indistinguishable. By definition, these colinear vectors represent the same point $[\psi] = [\psi']$ in the projective space $\mathbb{P}(\mathcal{H}_{d_1,\dots,d_n})$, where $\mathbb{P}(\mathcal{H})$ denotes the projective space of the Hilbert space $\mathcal{H}$. Therefore, it is natural to work in projective space when analyzing quantum states.

We first introduce the core variety of the stratification, which is the variety of separable states.

Definition 1.1. Let $\mathcal{H} = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$. Consider the following map:

$$\begin{aligned} \mathrm{Seg}: \quad & \mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-1} \times \cdots \times \mathbb{P}^{d_n-1} \to \mathbb{P}\left(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}\right) \\ & ([\varphi_1], [\varphi_2], \dots, [\varphi_n]) \mapsto [\psi] = [\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n]. \end{aligned} \tag{1.1}$$

The image of this map is called the **Segre variety** or the **Segre embedding** of $\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-1} \times \cdots \times \mathbb{P}^{d_n-1}$.

From the definition, the following properties are equivalent:

$$\begin{aligned} X = \mathrm{Seg}\left(\mathbb{P}^{d_1-1}\times\mathbb{P}^{d_2-1}\times\cdots\times\mathbb{P}^{d_n-1}\right) &\Leftrightarrow \quad X \text{ is the variety of separable states}\\ &\Leftrightarrow \quad X \text{ is the variety of rank-one tensors}\\ &\Leftrightarrow \quad X = \mathbb{P}\left(\mathrm{SLOCC}\,|0\ldots 0\rangle\right). \end{aligned}$$

Example 1. Consider the case $n = 2$ with $d_1 = d_2 = 2$. In this case, a two-qubit quantum state is represented by a normalized tensor in $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$, i.e., a tensor $|\psi\rangle$ such that

$$|\psi\rangle = a_{00}\,|00\rangle + a_{01}\,|01\rangle + a_{10}\,|10\rangle + a_{11}\,|11\rangle\,,$$

with the normalization condition $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$.

Using the isomorphism between a vector space and its dual, we have $\mathbb{C}^2\otimes\mathbb{C}^2 \cong \mathbb{C}^2\otimes(\mathbb{C}^2)^* = \mathcal{M}_2(\mathbb{C})$. Thus, in this particular case of a bipartite quantum state, one can also describe the tensor as a 2×2 matrix:

$$|\psi\rangle = \begin{pmatrix} a_{00} & a_{01}\\ a_{10} & a_{11}\end{pmatrix}.$$

Thus, $|\psi\rangle$ is separable if and only if it is a rank-one (matrix) tensor, i.e., $a_{00}a_{11} - a_{01}a_{10} = 0$. In other words, $[\psi] = [a_{00} : a_{01} : a_{10} : a_{11}] \in \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ if and only if $\det_{22}(\psi) = a_{00}a_{11} - a_{10}a_{01} = 0$. In the two-qubit case, the set of separable states is a projective variety of dimension 2 in $\mathbb{P}^3$ given by a quadratic equation, i.e., a quadric surface. ♢

Example 2. Another important example for these lectures is the three-qubit case, $n = 3$ and $d = 2$. In this case, the projectivized Hilbert space is seven-dimensional, $\mathbb{P}^7 = \mathbb{P}(\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2)$. The variety of separable states is given by the projectivization of the rank-one tensors, which is described by the Segre embedding:

$$\begin{aligned} \mathrm{Seg}: \quad &\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1 \to \mathbb{P}\left(\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\right)\\ &([\varphi_1],[\varphi_2],[\varphi_3]) \mapsto [\psi] = [\varphi_1\otimes\varphi_2\otimes\varphi_3]. \end{aligned} \tag{1.2}$$

Let us find the defining equations of $\mathrm{Seg}\,(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1) \subset \mathbb{P}^7$, i.e., the variety of separable states. Assume $|\psi\rangle = |\varphi_1\rangle\otimes|\varphi_2\rangle\otimes|\varphi_3\rangle$ with $|\varphi_1\rangle = a_0\,|0\rangle + a_1\,|1\rangle$, $|\varphi_2\rangle = b_0\,|0\rangle + b_1\,|1\rangle$, and $|\varphi_3\rangle = c_0\,|0\rangle + c_1\,|1\rangle$. Then

$$|\psi\rangle = a_0b_0c_0\,|000\rangle + a_0b_0c_1\,|001\rangle + \cdots + a_1b_1c_1\,|111\rangle\,,$$

i.e., in coordinates, we have:

$$\mathrm{Seg}\,([a_0 : a_1],[b_0 : b_1],[c_0 : c_1]) = [a_0b_0c_0 : a_0b_0c_1 : a_0b_1c_0 : a_0b_1c_1 : a_1b_0c_0 : a_1b_0c_1 : a_1b_1c_0 : a_1b_1c_1].$$

This set can be described by the following set of homogeneous equations:

- $Z_0Z_5 - Z_1Z_4 = 0, \quad Z_0Z_6 - Z_2Z_4 = 0, \quad Z_0Z_7 - Z_3Z_4 = 0,$
- $Z_0Z_3 - Z_1Z_2 = 0, \quad Z_0Z_6 - Z_2Z_4 = 0, \quad Z_0Z_7 - Z_2Z_5 = 0,$
- $Z_0Z_3 - Z_1Z_2 = 0, \quad Z_0Z_5 - Z_1Z_4 = 0, \quad Z_0Z_7 - Z_1Z_6 = 0.$

These equations are obtained from the 2×2 minors of the three flattenings of the tensor $|\psi\rangle = \sum a_{ijk}\,|ijk\rangle$:

$$\varphi^1 = \begin{pmatrix} a_{000} & a_{001} & a_{010} & a_{011}\\ a_{100} & a_{101} & a_{110} & a_{111}\end{pmatrix}, \quad \varphi^2 = \begin{pmatrix} a_{000} & a_{001} & a_{100} & a_{101}\\ a_{010} & a_{011} & a_{110} & a_{111}\end{pmatrix}, \quad \varphi^3 = \begin{pmatrix} a_{000} & a_{010} & a_{100} & a_{110}\\ a_{001} & a_{011} & a_{101} & a_{111}\end{pmatrix}.$$

One can show that $|\psi\rangle$ is of rank one if and only if all the matrices φ^i are of rank one. ♢

Remark 2. A tensor is of rank one if and only if all its $d_i \times d_1 \dots \hat{d_i} \dots d_n i$ flattenings are of rank one. This shows, using the 2×2 minors of the corresponding matrices, that $\mathrm{Seg}\left(\mathbb{P}^{d_1-1} \times \mathbb{P}^{d_2-1} \times \cdots \times \mathbb{P}^{d_n-1}\right)$ is an algebraic variety.

We now introduce auxiliary varieties that can be built from our core variety of separable states.

Definition 1.2 (Secant Variety)**.** Let $[x_1], \dots, [x_k]$ be k points of $\mathbb{P}(V)$. The $(k-1)$-plane $\mathbb{P}^{k-1}_{x_1 \dots x_k}$ is the projectivization of the linear span $\langle x_1, \dots, x_k \rangle \subset V$. We define the ***k*-secant variety** of $X \subset \mathbb{P}(V)$ as the Zariski closure of the union of the secant $(k-1)$-planes of X:

$$\sigma_k(X) = \overline{\bigcup_{[x_i] \in X} \mathbb{P}^{k-1}_{x_1 \dots x_k}}.$$

For $k = 2$, one often refers to $\sigma(X) = \sigma_2(X)$ as the secant variety.

The expected dimension of the k-secant variety is $\dim(\sigma_k(X)) = k \cdot d + (k-1)$, where $d = \dim(X)$. When this is not the case, we say that $\sigma_k(X)$ is **defective**, and we define the **secant defect** as $\delta_k = k \cdot d + (k-1) - \dim(\sigma_k(X))$.

Example 3. Let $X = \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8$, the variety of separable states of a two-qutrit system. The variety $\sigma_2(X)$ is precisely the projectivization of the 3×3 matrices of rank at most 2 in $\mathbb{P}^8 = \mathbb{P}(\mathcal{M}_{3\times 3}(\mathbb{C}))$. The third secant variety $\sigma_3(X)$ coincides with the closure of the projectivization of rank 3 matrices; that is, $\sigma_3(X) = \mathbb{P}^8$. Note that $\sigma_2(X)$ is a hypersurface of dimension 7 and thus is defective.

♢

Definition 1.3 (Tangential Variety)**.** Let $X \subset \mathbb{P}(V)$ be a smooth algebraic variety. Then the **tangential variety** of X is the union of its tangent spaces:

$$\tau(X) = \bigcup_{x \in X} T_x X.$$

By construction, we have $\tau(X) \subseteq \sigma(X)$.

The following theorem is a consequence of a result by Fulton and Hansen [14], later generalized by Zak [47]:

Theorem 1.4. *Let $X \subset \mathbb{P}(V)$ be a variety of dimension d. Then either:*

(1) $\dim(\tau(X)) = 2d$ *and* $\dim(\sigma(X)) = 2d + 1$*; or*
(2) $\tau(X) = \sigma(X)$*.*

Our last auxiliary variety belongs to the dual projective space.

Definition 1.5 (Dual Variety)**.** Let $X \subset \mathbb{P}(V)$ be a projective algebraic variety. We define the **dual variety** of X as

$$X^* = \overline{\{H \in \mathbb{P}(V^*) \mid \exists x \in X_{\text{smooth}},\ T_x X \subset H\}}.$$

One expects the dual variety to be a hypersurface; when it is, we denote its defining equation by Δ_X. If X^* is not a hypersurface, we define its **dual defect** as $\delta_{X^*} = \dim(\mathbb{P}(V)) - 1 - \dim(X^*)$.

Example 4. Returning to the two-qutrit example, let $X = \mathrm{Seg}(\mathbb{P}^2 \times \mathbb{P}^2) \subset \mathbb{P}^8 = \mathbb{P}(\mathcal{M}_{3\times 3}(\mathbb{C}))$. We identify $\mathcal{H}_{3,3} = \mathbb{C}^3 \otimes \mathbb{C}^3$ with the space of 3×3 complex matrices and consider its dual space accordingly. Without loss of generality, since X is homogeneous under the action of $\mathrm{SL}_3 \times \mathrm{SL}_3$, we consider $[\psi] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ as a representative point of X.

The tangent space at $[\psi]$ is given by

$$T_{[\psi]}X = \mathbb{P}\left(\left\{\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}\right\}\right).$$

This is evident if we apply the chain rule to differentiate at $t = 0$ the curve in X defined by $[\psi(t)] = [\psi_1(t) \otimes \psi_2(t)]$, where $|\psi_1(0)\rangle = |\psi_2(0)\rangle = |0\rangle$.

Then, a matrix H is tangent to $[\psi]$ if and only if

$$H = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

i.e., if and only if $\det(H) = 0$.

$\diamondsuit$

1.1.1. *General Facts about Auxiliary Varieties of* $X = Seg\left(\mathbb{P}^{d_1-1} \times \cdots \times \mathbb{P}^{d_n-1}\right) \subset \mathbb{P}^{d_1 d_2 \dots d_n - 1}$. Here are some well-known general facts in the case when X is our set of separable states:

- $\sigma_k(X)$, $\tau(X)$, X^*, and $\mathrm{Sing}(X^*)$ are SLOCC-invariant varieties.
- $\sigma_k(X)$ is the closure of the set of tensors of rank at most k.
- For $d_1 = \cdots = d_n = 2$, and $n \geq 5$, $\sigma_k(X)$ is not defective for all k [20].
- $\sigma(X) = \mathbb{P}(\overline{\mathrm{SLOCC}\,(|0\dots0\rangle + |1\dots1\rangle)}) = \mathbb{P}(\overline{\mathrm{SLOCC}\,|GHZ_n\rangle})$.
- $\tau(X) = \overline{\mathrm{SLOCC}\cdot(|10\dots0\rangle + |010\dots0\rangle + \cdots + |00\dots1\rangle)} = \mathbb{P}(\overline{\mathrm{SLOCC}\cdot|W_n\rangle})$.
- If $n = 2$ and $d = d_1 = d_2$, Δ_X is the determinant of $d \times d$ matrices. If $n \geq 2$ and X^* is a hypersurface, Δ_X is called the Hyperdeterminant [16] of format $d_1 \times d_2 \times \cdots \times d_n$. We will denote it by $\mathrm{HDet}_{d_1 \dots d_n}$.
- X^* is a hypersurface if and only if $d_j - 1 \leq \sum_{i \neq j}(d_i - 1)$ for all j (see [16]).

Using the natural pairing between $\mathcal{H}_{d_1\dots d_n}$ and its dual, one considers that $[\psi] \in X^* \subset \mathbb{P}\left(\mathcal{H}_{d_1\dots d_n}\right)$ if and only if the corresponding linear form $\langle\psi|$, when restricted to $X \subset \mathbb{P}^{d_1 d_2 \dots d_n - 1}$, is singular.

Therefore, $[\psi] \in X^*$ for $|\psi\rangle = \sum_{(i_1,i_2,\dots,i_n)\in I} a_{i_1 i_2 \dots i_n} |i_1 i_2 \dots i_n\rangle$ if and only if the multilinear form

$$f\left(x^{(1)}, \dots, x^{(n)}\right) = \sum_{(i_1,i_2,\dots,i_n)\in I} a_{i_1 i_2 \dots i_n} x^{(1)}_{i_1} x^{(2)}_{i_2} \dots x^{(n)}_{i_n}$$

is singular.

The idea of using auxiliary varieties to provide a stratification of the ambient space probably goes back to the 19$^{\text{th}}$ century. In the quantum information literature, the idea of using these auxiliary varieties to describe entanglement classes first appeared in papers by Heydari [21, 22] for the use of secant varieties and in a series of papers by Miyake for the use of hyperdeterminants [36, 37]. This natural idea has also been explored by other authors, such as [41, 43] and more recently by Masoud Gharahi [17–19].

Example 5. The $2 \times 2 \times 2$ hyperdeterminant is

$$\begin{aligned}\mathrm{HDet}_{222}(A) = {} & a_{000}^2 a_{111}^2 + a_{010}^2 a_{101}^2 + a_{001}^2 a_{110}^2 + a_{011}^2 a_{100}^2 \\ & + 4\left(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111}\right) \\ & - 2\left(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111}\right. \\ & \quad \left. + a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{100}a_{110} + a_{010}a_{011}a_{100}a_{101}\right).\end{aligned}$$

There is a combinatorial picture associated with this polynomial [4]. Consider the cube in Figure 1 labeled by the entries of the $2 \times 2 \times 2$ tensor A. The three groups of monomials in $\mathrm{HDet}_{222}(A)$ are derived from the diagonals, parallelograms, and tetrahedras inside the cube.

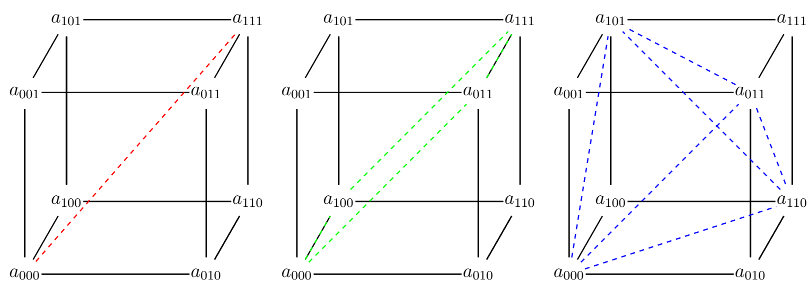

FIGURE 1. Monomials of Cayley's hyperdeterminant from a combinatorial perspective: The four diagonals (red) of the cube provide the first four monomials of degree 4 of type $a_{ijk}a_{\overline{ijk}}$ (where the bar denotes bit-complement), the six parallelograms (green) provide the six monomials of type $a_{ijk}a_{\overline{ijk}}a_{i'j'k'}a_{\overline{i'j'k'}}$ (where ijk and $i'j'k'$ differ in one bit), and the two tetrahedra (blue) give the two monomials of type $a_{ijk}a_{i\overline{jk}}a_{\bar{i}j\bar{k}}a_{\overline{ij}k}$.

◊

1.2. **The Three-Qubit Classification.** Up to SLOCC transformations, there are six classes of entanglement in the three-qubit case:

- $|000\rangle$ (separable states),
- $|B_1\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |011\rangle)$ (biseparable),
- $|B_2\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |101\rangle)$ (biseparable),
- $|B_3\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |110\rangle)$ (biseparable),
- $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ (fully entangled),
- $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ (fully entangled).

This classification gained significant attention when it was rediscovered in the quantum information literature [11] in the early 2000s because it explicitly provided two examples of fully entangled states that are not equivalent in terms of quantum resources.

Using the language of auxiliary varieties, one can provide the following descriptions:

- Employing the notions of secant and tangential varieties, a straightforward calculation shows that $\dim\left(\sigma\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right)\right) = 7$, which is both the expected dimension and the dimension of the ambient space. Therefore, by Theorem 1.4, the first secant fills the ambient space, and there exists a codimension-one subvariety corresponding to $\tau\left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right)$. The interpretation of the secant and tangential varieties imply the existence of the GHZ and W orbits.

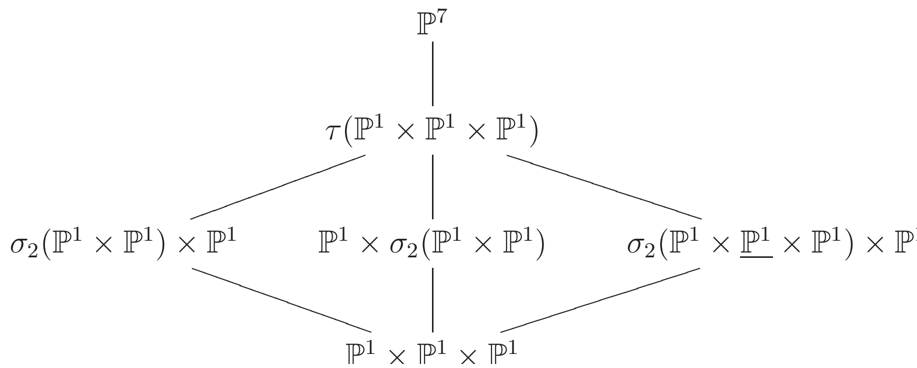

FIGURE 2. Description of the three-qubit entanglement classes by secant and tangential varieties.

- Using the dual perspective, one knows that $X^* = \left(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\right)^*$ is a hypersurface defined by Cayley's hyperdeterminant. One also needs to characterize the singular locus of X^*. When X^* is a hypersurface, smooth points of X^* correspond to hyperplane sections with a unique singular point of Morse type (i.e., with a Hessian of full rank). Thus, a hyperplane section can be singular if the Hessian is not of full rank (the cusp component of the singular locus) or if the hyperplane section has more than one singular point (the node set). In what follows, $X^*_{\text{node}}(\{1\})$ (resp. $X^*_{\text{node}}(\{2\})$ and $X^*_{\text{node}}(\{3\})$) is the Zariski closure of the hyperplanes tangent to two points $[\varphi]$ and $[\varphi']$ of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that $[\varphi] = [\varphi_1 \otimes \varphi_2 \otimes \varphi_3]$ and $[\varphi'] = [\varphi_1 \otimes \varphi_2' \otimes \varphi_3']$ (respectively for $\{2\}$ and $\{3\}$).

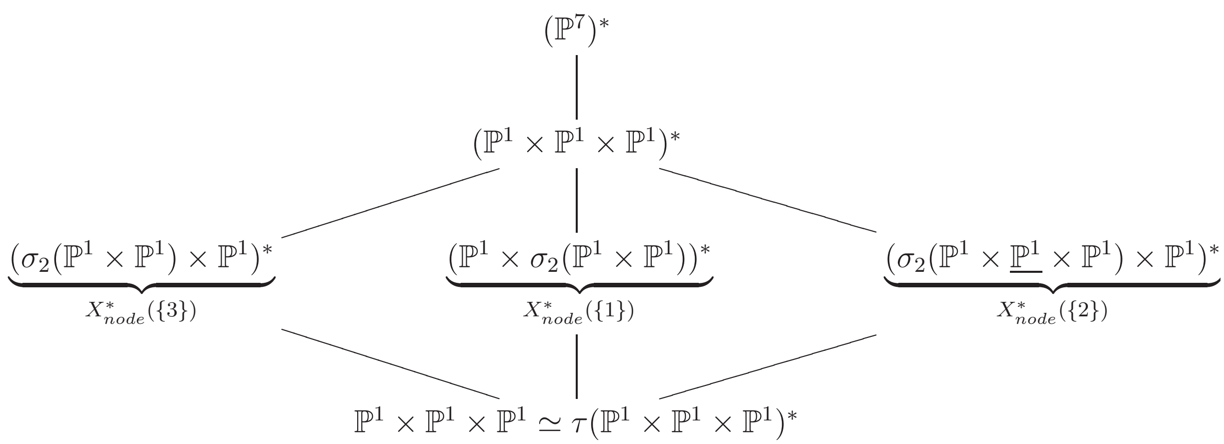

FIGURE 3. Description of entanglement classes in the three-qubit case by dual varieties.

One deduces Algorithm 1 to identify to which stratum a given state belongs.

In the same spirit, in [25], we similarly provided geometric descriptions and algorithms for $(d_1 = d_2 = 2, d_3 \geq 3)$ as well as $d_1 = 2$, $d_2 = d_3 = 3$.

Algorithm 1 Classification of Three-Qubit States via Hyperdeterminant and Flattening

Require: Three-qubit state $|\psi\rangle$ represented by tensor a_{ijk} with $i, j, k \in \{0, 1\}$
Ensure: Classification of $|\psi\rangle$ as GHZ-like, W, biseparable, or separable

```
Compute the hyperdeterminant HDet_222(A) of A = (a_ijk)
if HDet_222(A) ≠ 0 then
    return |ψ⟩ is SLOCC equivalent to GHZ
else
    Flatten a_ijk along each bipartition to obtain matrices:
    φ^1_{A|BC} ← reshape a_ijk into a 2 × 4 matrix with indices (i, (jk))
    φ^2_{B|AC} ← reshape a_ijk into a 2 × 4 matrix with indices (j, (ik))
    φ^3_{C|AB} ← reshape a_ijk into a 2 × 4 matrix with indices (k, (ij))
    Compute the ranks r_1 = rank(φ^1_{A|BC}), r_2 = rank(φ^2_{B|AC}), r_3 = rank(φ^3_{C|AB})
    if r_1 = r_2 = r_3 = 2 then
        return |ψ⟩ is SLOCC equivalent to W state
    else if Exactly one of r_1, r_2, or r_3 is 1 then
        return |ψ⟩ is biseparable
    else
        return |ψ⟩ is separable
    end if
end if
```

1.3. **Tripartite Quantum Systems with $|W\rangle$ and $|GHZ\rangle$-like Genuine Entangled States.** The formalism and geometric approach developed for multiqudit Hilbert spaces apply to other types of quantum systems, such as bosons or fermions, i.e. quantum states with symmetries. If we change the nature of the Hilbert space under consideration, we also need to adjust the SLOCC group of transformations accordingly. Let us mention two other well-known quantum systems:

Example 6. Bosons are indistinguishable particles that are symmetric under exchange. Thus, the corresponding Hilbert space is the space of symmetric tensors. If we consider bosonic qubits, we have:

- $\mathcal{H} = \mathrm{Sym}^n(\mathbb{C}^2)$, the symmetric subspace within $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$,
- $\mathrm{SLOCC} = SL_2(\mathbb{C})$,
- The variety of separable states is the Veronese curve:

$$v_n : \begin{cases} \mathbb{P}^1 \to \mathbb{P}^n = \mathbb{P}\left(\mathrm{Sym}^n(\mathbb{C}^2)\right), \\ [\varphi] \mapsto [\varphi \circ \varphi \circ \cdots \circ \varphi], \end{cases}$$

where $\circ$ denotes the symmetric tensor product.

$\diamondsuit$

Example 7. Fermions are indistinguishable particles that are skewsymmetric under exchange. Therefore, the corresponding Hilbert space is the space of skewsymmetric tensors. Consider k fermions where each particle is an n-dit:

- $\mathcal{H} = \bigwedge^k(\mathbb{C}^n)$, the skewsymmetric subspace within $\underbrace{\mathbb{C}^n \otimes \cdots \otimes \mathbb{C}^n}_{k \text{ times}}$,
- $\mathrm{SLOCC} = SL_n(\mathbb{C})$,

- The variety of separable states is the Grassmannian variety $G(k,n)$ of k-planes in $\mathbb{C}^n$:

$$v: \quad G(k,n) \hookrightarrow \mathbb{P}\left(\bigwedge^k \mathbb{C}^n\right),$$
$$[\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k] \mapsto [\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k]\,.$$

$\diamondsuit$

One way to rephrase the results of [11] regarding the two inequivalent types of entanglement for three-qubit quantum states is to observe again that, in the three-qubit case, the secant variety fills the ambient space and has the expected dimension. This accounts for the existence of the two distinguished orbits corresponding to the $|GHZ\rangle$ and $|W\rangle$ states.

This raises the question of whether there are other types of Hilbert spaces $\mathcal{H}$ and corresponding SLOCC groups such that $\tau(\text{SLOCC}\cdot[\varphi]) \neq \sigma(\text{SLOCC}\cdot[\varphi]) = \mathbb{P}(\mathcal{H})$, where $[\varphi]$ is a separable state. The answer, as provided in [24], is summarized in the following table:

G	$\mathcal{H}$	variety (orbit) of separable states	QIT interpretation	References
$SL_2(\mathbb{C})$	$Sym^3(\mathbb{C}^2)$	$v_3(\mathbb{P}^1) \subset \mathbb{P}^3$	Three bosonic qubits	Brody, Gustavsson, Hughston[6] Vrana and Lévay[46]
$SL_2(\mathbb{C}) \times SO(m)$				
$m=3$	$\mathbb{C}^2 \otimes Sym^2(\mathbb{C}^2)$	$\mathbb{P}^1 \times v_2(\mathbb{P}^1) \subset \mathbb{P}^5$	1 distinguished qubit and 2 bosonic qubits	Vrana and Lévay[46]
$m=4$	$\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$	3 qubits	Dür, Vidal, Cirac[11]
$m=5$	$\mathbb{C}^2 \otimes \Lambda^{<2>}\mathbb{C}^4$	$\mathbb{P}^1 \times LG(2,4) \subset \mathbb{P}^9$ LG denotes the Lagrangian Grassmanian	1 distinguished qubit and two fermions with 4 single-particle state and a symplectic form condition	New
$m=6$	$\mathbb{C}^2 \otimes \Lambda^2\mathbb{C}^4$	$\mathbb{P}^1 \times G(2,4) \subset \mathbb{P}^{11}$	1 qubit and two fermions with 4 single-particle states	Vrana and Lévay[46]
$m>6$	$\mathbb{C}^2 \otimes \mathbb{C}^m$	$\mathbb{P}^1 \times Q^{m-2} \subset \mathbb{P}^{2m-1}$	1 qubit and 1 isotropic (m-1)-dits	New
$Sp_6(\mathbb{C})$	$\Lambda^{<3>}\mathbb{C}^6$	$LG(3,6) \subset \mathbb{P}^{13}$	Three fermions with six single-particle states and a symplectic form condition	New
$SL_6(\mathbb{C})$	$\Lambda^3\mathbb{C}^6$	$G(3,6) \subset \mathbb{P}^{19}$	Three fermions with six single-particle states	Levay and Vrana[46]
$Spin_{12}$	Δ_{12}	$\mathbb{S}_6 \subset \mathbb{P}^{31}$ Spinor variety	Particles in Fermionic Fock spaces	Sárosi and Lévay[42]
E_7	V_{56}	$E_7/P_1 \subset \mathbb{P}^{55}$ unique closed orbit of the fund. rep. of E_7	Tripartite entanglement of seven qubits	Duff and Ferrara[10]

TABLE 1. Classification of smooth G-orbits satisfying $\tau(G.[v]) \subsetneq \sigma(G.[v]) = \mathbb{P}(\mathcal{H})$

Remark 3. Table 1 is deduced from the classification of irreducible representations $(\mathcal{H}, G)$ such that the ring of invariant is generated by a unique invariant polynomial $\mathbb{C}[\mathcal{H}]^G = \mathbb{C}[F]$. The table is then obtained by considering the representations such that $\sigma(X) = \mathbb{P}(\mathcal{H})$ and $2\dim(X) + 1 = \dim(\mathcal{H}) - 1$ where X is the highest weight orbit of the representation. Then F is the defining equation of the dual of X.

2. Invariant Theory, Discriminants, and Hyperdeterminants

2.1. **Classification of Binary Forms and the Entanglement of Symmetric Qubit States.** The study of binary forms is a classical subject in invariant theory dating back to the early 19$^{\text{th}}$ century. Many algorithms, classifications, and early computations were developed by the British school led by Cayley and Sylvester. Their techniques have regained attention at the end of the 20$^{\text{th}}$ century with the advent of computer algebra systems. In what follows, we will use [39] as our main reference.

To quote Olver, classical invariant theory is the study of the "intrinsic or geometrical properties of polynomials."

Example 8. Consider a degree-two polynomial in two variables, i.e., a binary form of degree 2:

$$P(x, y) = ax^2 + 2bxy + cy^2.$$

It is well-known that the discriminant $\Delta_2(P) = b^2 - ac$ is invariant under $SL_2(\mathbb{C})$ changes of basis.

Using the geometric perspective developed in Section 1, the space of binary forms of degree two is a three-dimensional vector space spanned by x^2, $2xy$, and y^2, or equivalently, the space of 2×2 symmetric matrices over $\mathbb{C}^2$ or the space of two bosonic qubits, denoted $Sym^2(\mathbb{C}^2)$. The Veronese map parametrizes the space of rank-one symmetric tensors, or in other words, degree-two polynomials that correspond to the squares of linear forms:

$$v_2 : \begin{cases} \mathbb{P}(\mathbb{C}^2) \to \mathbb{P}\left(Sym^2(\mathbb{C}^2)\right), \\ [x : y] \mapsto [x^2 : 2xy : y^2]. \end{cases}$$

Then the discriminant is nothing but the equation of the dual variety of $v_2(\mathbb{P}^1)$. Note that Δ_2 is also the restriction of the usual determinant of 2×2 matrices to the space of symmetric matrices.

Invariants	(Canonical) Binary Form	Symmetric States	Geometry
$\Delta_2 \neq 0$	$x^2 + y^2$	$\|00\rangle + \|11\rangle$	$\mathbb{P}^2 \setminus \{\Delta = 0\}$
$\Delta_2 = 0$	x^2	$\|00\rangle$	$v_2(\mathbb{P}^1) \subset \mathbb{P}^2$

TABLE 2. Classification of degree-two binary forms and symmetric two-qubit states.

♢

The classification of symmetric three-qubit states similarly follows from the knowledge of the discriminant of cubic binary forms.

Example 9. Let us consider $f(x, y) = a_0x^3 + 3a_1x^2y + 3a_2xy^2 + a_3y^3$; then

$$\Delta_3 = a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 - 3a_1^2a_2^2 + 4a_1^3a_3.$$

This leads to the following classification of symmetric three-qubit states (binary forms of degree 3):

Invariants	(Canonical) Binary Form	Symmetric States	Geometry
$\Delta_3 \neq 0$	$x^3 + y^3$	$\|000\rangle + \|111\rangle$	$\mathbb{P}^3 \setminus \{\Delta = 0\}$
$\Delta_3 = 0$	x^2y	$\|001\rangle + \|010\rangle + \|100\rangle$	$v_3(\mathbb{P}^1)^*$
2×4 flattenings have rank one	x^3	$\|000\rangle$	$v_3(\mathbb{P}^1)$

TABLE 3. Classification of degree-three binary forms and symmetric three-qubit states.

♢

In these two examples, the ring of invariant polynomials is generated by the discriminant.

Classical invariant theory of binary forms deals not only with finding invariants but also with covariant polynomials of binary forms to classify them. Recall that a *covariant polynomial* is a polynomial in the coefficients of the form and the auxiliary variables x and y, which is invariant under the action of $SL_2(\mathbb{C})$ on both the coefficients and the variables. More precisely, if $f(x,y)=\sum_{k=0}^{n}\binom{n}{k}a_kx^{n-k}y^k$ is a binary form and $g\in SL_2(\mathbb{C})$ acts on f as $g\cdot f(x,y)=\sum_k a_k(ax+by)^{n-k}(cx+dy)^k=\sum_k \tilde{a}_kx^{n-k}y^k$, then P is a covariant polynomial if and only if

$$P(a_0,\dots,a_n,x,y)=P(\tilde{a}_0,\dots,\tilde{a}_n,dx-cy,-bx+ay).$$

For instance, the *ground form* is a covariant polynomial:

$$f(x,y)=\sum_k\binom{n}{k}a_kx^{n-k}y^k. \tag{2.1}$$

An *invariant* of the form is a covariant polynomial that depends only on the coefficients.

Let us now consider the spaces of binary forms of degrees p and q, $\mathrm{Sym}^p\,\mathbb{C}^{2*}$ and $\mathrm{Sym}^q\,\mathbb{C}^{2*}$. The *transvection* operation of degree d on $\mathrm{Sym}^p\,\mathbb{C}^{2*}\times\mathrm{Sym}^q\,\mathbb{C}^{2*}$ is defined by

$$(f,g)_d=\sum_{j=1}^{d}(-1)^j\binom{d}{j}\frac{\partial^d f}{\partial x^{d-j}\partial y^j}\frac{\partial^d g}{\partial x^j\partial y^{d-j}}, \tag{2.2}$$

where $f\in\mathrm{Sym}^p\,\mathbb{C}^{2*}$ and $g\in\mathrm{Sym}^q\,\mathbb{C}^{2*}$.

Starting from the ground form and repeatedly applying the transvection process, Gordan's theorem [39] ensures that all covariant and invariant polynomials of binary forms of degree n can be generated.

For example, for $n=4$, it is well-known that the ring of invariant polynomials of binary quartics is generated by two polynomials I_2 and I_3:

$$\mathbb{C}\left[\mathrm{Sym}^4(\mathbb{C}^{2*})\right]^{SL_2(\mathbb{C})}=\mathbb{C}[I_2,I_3], \tag{2.3}$$

where

$$I_2=a_0a_4-4a_1a_3+3a_2^2$$

and

$$I_3=\begin{vmatrix}a_0 & a_1 & a_2\\ a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\end{vmatrix}.$$

Then the discriminant is given by $\Delta_4=I_2^3-27I_3^2$. Two important covariants for binary quartics are the Hessian and its Jacobian:

$$\mathrm{Hess}(f)=\begin{vmatrix}\dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\partial y}\\ \dfrac{\partial^2 f}{\partial y\partial x} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix},\quad T(f)=\begin{vmatrix}\dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y}\\ \dfrac{\partial\,\mathrm{Hess}(f)}{\partial x} & \dfrac{\partial\,\mathrm{Hess}(f)}{\partial y}\end{vmatrix}. \tag{2.4}$$

The ground form f, the covariants $\mathrm{Hess}(f)$ and $T(f)$, and the invariants I_2 and I_3 generate all invariants and covariants of the space of binary quartics. The multiplicities of the roots of a quartic can be determined using these invariants, as shown in Table 4. This table leads to a classification of binary quartics [39, page 29].

Canonical Form	Invariants	Roots
$x^4+\lambda x^2y^2+y^4$	$\lambda\neq\pm 2,\ \Delta_4\neq 0$	4 simple roots
$x^2(x^2+y^2)$	$\Delta_4=0,\ T\neq 0$	One double root
x^2y^2	$\Delta_4=0,\ T=0,\ I_2\neq 0$	Two double roots
x^3y	$I_2=I_3=0,\ H\neq 0$	One triple root
x^4	$H=0,\ f\neq 0$	One quadruple root

TABLE 4. Classification of binary quartics [39, page 29].

Let us denote by for $k=0\dots 4$, the following basis of the space of symmetric four-qubit states (Dicke states):

$$|D_k\rangle=\frac{1}{\sqrt{\binom{4}{k}}}\sum_{\text{all permutations } P}|P(\underbrace{1\dots 1}_{k}\underbrace{0\dots 0}_{4-k})\rangle \tag{2.5}$$

This table translates to the following classification of symmetric four-qubit states:

Normal Form	SLOCC Algebraic Variety (Orbit Closure)	Invariants/Covariants
$\|D_0\rangle+\lambda\|D_2\rangle+\|D_4\rangle$	$\mathbb{P}^4\setminus X^*_{\text{Sep}}$	$\Delta_4\neq 0$
$\|D_0\rangle+i\sqrt{2}\|D_2\rangle+\|D_4\rangle$	$\mathcal{Q}_{\text{smooth}}$	$\Delta_4\neq 0, I_2=0$
$\|D_0\rangle+\|D_4\rangle$	$\sigma_2(X_{\text{Sep}})$	$\Delta_4\neq 0, I_3=0$
$\|D_0\rangle+\|D_2\rangle$	X^*_{Sep}	$\Delta_4=0$
$\|D_2\rangle$	$(X_{\text{Sep}})^*_{\text{node}}\simeq\sigma_2(X_{\text{Sep}})^*$	$\Delta_4=T=0,\ I_2\neq 0$
$\|D_1\rangle$	$(X_{\text{Sep}})^*_{\text{cusp}}\simeq\tau(X_{\text{Sep}})$	$I_2=I_3=0,\ H\neq 0$
$\|D_0\rangle$	$X_{\text{Sep}}=v_4(\mathbb{P}^1)$	$I_2=I_3=T=H=0$

TABLE 5. Symmetric four-qubit states classification: There exist three SLOCC hypersurfaces defined respectively by $I_2=0$, $I_3=0$, and $\Delta_4=0$, denoted by $\mathcal{Q}_{\text{smooth}}$ (the smooth quadric), $\sigma_2(X_{\text{Sep}})$ (the secant variety of the Veronese curve), and X^*_{Sep} (the dual variety of the Veronese curve). The node and cusp components of X^*_{Sep} correspond to the singular locus of the discriminant (see [36]).

A pictorial representation of the stratification given by Table 5 is provided in Figure 4.

2.2. **The Four-Qubit Hyperdeterminant from a Classical Invariant Perspective.** With my co-authors Jean-Gabriel Luque and Jean-Yves Thibon, we investigated the four-qubit classification using a combination of classical invariant theory and algebraic geometry [26, 27]. The four-qubit classification is a more challenging problem, as it is the first n-qubit system with an infinite number of orbits.

Here, $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$, and a four-qubit quantum state is a tensor/hypermatrix of format $2\times 2\times 2\times 2$:

$$|\psi\rangle=\sum_{i,j,k,l\in\{0,1\}}a_{ijkl}\,|ijkl\rangle\,.$$

The four-qubit algebra of invariant polynomials has been calculated in [34]. It is generated by four polynomials of degrees 2, 4, 4, and 6:

$$\mathbb{C}\left[\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2\right]^{\text{SLOCC}}=\mathbb{C}[H,L,M,D_{xy}]. \tag{2.6}$$

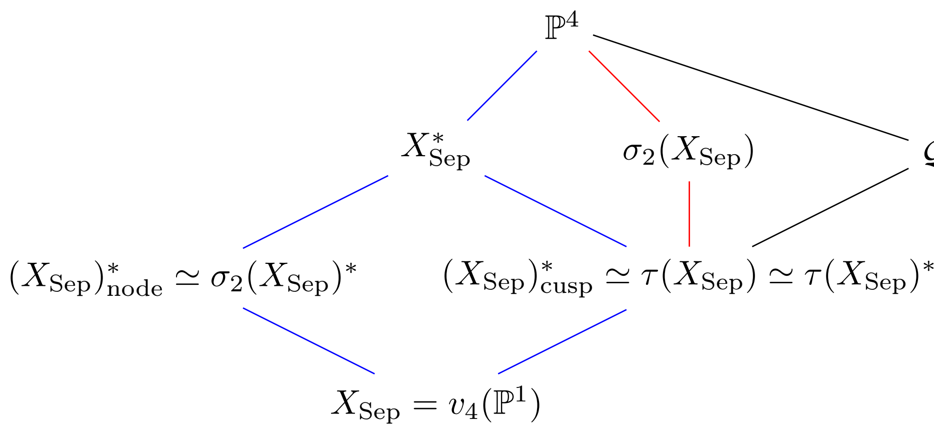

FIGURE 4. Stratification of the projectivized Hilbert space of symmetric four-qubit states by SLOCC-invariant algebraic varieties. The stratification induced by the inclusions corresponding to the blue edges is the classification provided by Aulbach [3] and can be interpreted as a discussion on the discriminant Δ_4. The stratification induced by the inclusions corresponding to the red edges corresponds to the stratification by secant varieties as provided by Sanz *et al.* [41].

The four generators H, L, M, and D_{xy} are defined as follows, with the ground form being $f = \sum_{i,j,k,l\in\{0,1\}} a_{ijkl}x_iy_jz_kt_l$.

(1) One invariant of degree 2:

$$H = a_{0000}a_{1111} - a_{1000}a_{0111} - a_{0100}a_{1011} + a_{1100}a_{0011} - a_{0010}a_{1101} + a_{1010}a_{0101} + a_{0110}a_{1001} - a_{1110}a_{0001}.$$

(2) Two invariants of degree 4:

$$L := \begin{vmatrix} a_{0000} & a_{0010} & a_{0001} & a_{0011} \\ a_{1000} & a_{1010} & a_{1001} & a_{1011} \\ a_{0100} & a_{0110} & a_{0101} & a_{0111} \\ a_{1100} & a_{1110} & a_{1101} & a_{1111} \end{vmatrix},$$

$$M := \begin{vmatrix} a_{0000} & a_{0001} & a_{0100} & a_{0101} \\ a_{1000} & a_{1001} & a_{1100} & a_{1101} \\ a_{0010} & a_{0011} & a_{0110} & a_{0111} \\ a_{1010} & a_{1011} & a_{1110} & a_{1111} \end{vmatrix}.$$

(3) One invariant of degree 6: Set $b_{xy} := \det\left(\dfrac{\partial^2 f}{\partial z_i \partial t_j}\right)$. This quadratic form is interpreted as a bilinear form on the three-dimensional space $\mathrm{Sym}^2(\mathbb{C}^2)$, so we can find a 3×3 matrix B_{xy} satisfying

$$b_{xy} = [x_0^2, x_0x_1, x_1^2]B_{xy}\begin{bmatrix} y_0^2 \\ y_0y_1 \\ y_1^2 \end{bmatrix}.$$

The generator of degree 6 is $D_{xy} := \det(B_{xy})$.

Note that we can alternatively replace L or M by

$$N := -L - M = \begin{vmatrix} a_{0000} & a_{1000} & a_{0001} & a_{1001} \\ a_{0100} & a_{1100} & a_{0101} & a_{1101} \\ a_{0010} & a_{1010} & a_{0011} & a_{1011} \\ a_{0110} & a_{1110} & a_{0111} & a_{1111} \end{vmatrix},$$

and D_{xy} by $D_{xz}, \ldots, D_{zt}$ defined in a similar way with respect to the variables $xz, \ldots, zt$ (see [34]).

In the four-qubit case, the hyperdeterminant HDet_{2222} is a degree 24 invariant [16]. Let us consider a four-qubit state $|\psi\rangle$ and decompose it according to the last qubit:

$$|\psi\rangle = |\varphi_1\rangle \otimes |0\rangle + |\varphi_2\rangle \otimes |1\rangle,$$

where $|\varphi_1\rangle$ and $|\varphi_2\rangle$ are three-qubit states. Then, a general method of Schläfli allows us to show in the four-qubit case that:

$$\mathrm{HDet}_{2222}(\psi) = \Delta\left(\mathrm{HDet}_{222}(x\varphi_1 + y\varphi_2)\right),$$

where Δ is the discriminant of a quartic binary form.

Let us consider the quartic $Q(x, y) = \mathrm{HDet}_{222}(x\varphi_1 + y\varphi_2) = a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4$. As we saw, the algebra of invariants of binary quartics is generated by

$$S = a_0a_4 - 4a_1a_3 + 3a_2^2, \quad T = a_0a_2a_4 - a_0a_3^2 + 2a_1a_2a_3 - a_1^2a_4 - a_2^3.$$

Then, an alternative expression of HDet_{2222} is

$$\mathrm{HDet}_{2222} = S^3 - 27T^2.$$

The polynomials S and T are SLOCC invariants (for an expression in terms of H, L, M, D_{xy}, see [34]); the quartic $Q(x, y)$ is a covariant.

Using similar classical invariant techniques, Briand, Luque, and Thibon [5] computed a complete set of covariants in the four-qubit case. In [26], we used this set to provide an algorithm that separates the orbits of the null cone of the Hilbert space, i.e., the variety corresponding to the zero locus of all polynomial invariants. In [27], we used different expressions of $R(x, y)$ to provide an algorithm to identify a quantum state and its canonical form provided by Verstraete's classification [8, 45].

The Four-Qubit Hyperdeterminant from Projection of G-Discriminant. I now propose an alternative way of computing HDet_{2222} as the restriction of the defining equation of a higher-dimensional discriminant. The basic idea is the following geometric argument proved with my coauthor Luke Oeding in [28]:

Theorem 2.1 (Tangency Condition [28])**.** *Consider a non-trivial vector space splitting $V = A \oplus B$.*

Let $X \subset \mathbb{P}V$ and $Y \subset \mathbb{P}A$ be projective varieties. Let π_B denote the rational map $\mathbb{P}V \dashrightarrow \mathbb{P}A$ induced from the projection $V \to A$.

If for each smooth point $[y] \in Y$ there is a smooth point $[x] \in X$ such that $\pi_B(\widehat{T}_xX) \subset \widehat{T}_yY$, then

$$Y^* \subseteq X^* \cap \mathbb{P}A^*.$$

Moreover, if X^* and Y^* are hypersurfaces defined respectively by polynomials Δ_X and Δ_Y, and for every general point $[h] \in Y^*$, $H = h^\perp$, viewed as a hyperplane in $\mathbb{P}V$, is a point of multiplicity m of X^*, then

$$\Delta_Y^m \mid \mathrm{Res}(\Delta_X, A^*).$$

Our tangency condition extends a result of [16] and guarantees that the restriction of a discriminant is divisible by the equation of another dual variety.

Our main application is when V is a simple Lie algebra.

Recall that for $\mathfrak{g}$ a semisimple Lie algebra, the *adjoint variety*, denoted X_G^{ad}, is the projectivization of the highest weight orbit in $\mathfrak{g}$ for the adjoint action of the Lie group G,

$$X_G^{\text{ad}} = \mathbb{P}(G \cdot v) \subset \mathbb{P}\mathfrak{g}, \tag{2.7}$$

with v a highest weight vector of $\mathfrak{g}$.

The adjoint variety X_G^{ad} is the unique closed orbit for the adjoint action on $\mathbb{P}\mathfrak{g}$.

More generally, suppose V_λ is an irreducible G-module with highest weight λ, and highest weight vector $v_\lambda \in V_\lambda$. Consider the *homogeneous variety* $G \cdot v_\lambda = G/P$, where P is the stabilizer of $v_\lambda \in V_\lambda$. Then [15, Claim 23.52] says that G/P is the unique closed orbit of G acting on V_λ.

Duals of adjoint varieties are hypersurfaces [44], and as such, are defined by a single (up to scale) irreducible homogeneous polynomial, the *G-discriminant*, denoted Δ_G (instead of $\Delta_{X_G^{\text{ad}}}$).

Suppose $\mathfrak{g}$ is equipped with a $\mathbb{Z}_k$-grading

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{k-1}, \tag{2.8}$$

and $\mathfrak{g}_0$ is such that $\exp(\mathfrak{g}_0) =: G_0$ is a connected component containing the identity of G, so that G_0 is the (Lie) subgroup of G with Lie algebra $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ acts on $\mathfrak{g}$ and preserves the grading (i.e., $[\mathfrak{g}_0, \mathfrak{g}_i] \subset \mathfrak{g}_i$), then (2.8) is a G_0-module decomposition. One can establish relations between duals of G_0-orbits and the restriction of the G-discriminant by Theorem 2.1.

Theorem 2.2. *Using the notation above, suppose $\mathbb{P}\mathfrak{g}_s \cap X_G^{ad} \neq \emptyset$ and let $[v_\lambda]$ be one such point of this intersection. The respective homogeneous varieties $Y = G_0 \cdot [v_\lambda] \subset \mathbb{P}\mathfrak{g}_s$ and $X_G^{ad} = G \cdot [v_\lambda] \subset \mathbb{P}\mathfrak{g}$ satisfy the tangency condition; hence, $Y^* \subset X_G^{ad*} \cap \mathbb{P}(\mathfrak{g}_s^*)$. If Y^* is a hypersurface, then $\Delta_Y^m \mid \operatorname{Res}(\Delta_{X_G^{ad}}, \mathfrak{g}_s)$, for some integer $m > 0$.*

In particular, let us consider $G = SO(8)$ and the following grading of the Lie algebra $\mathfrak{so}(8)$:

$$\mathfrak{so}(8) = \underbrace{\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2}_{=\mathfrak{g}_0} \oplus \underbrace{\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2}_{=\mathfrak{g}_1}. \tag{2.9}$$

The varieties $X_{SO(8)}^{\text{ad}}$ and $Y = \text{Seg}(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$ satisfy the conditions of Theorem 2.2, and thus HDet_{2222} divides the restriction of the discriminant of $SO(8)$.

Recall that $\mathfrak{so}(8) = \{A \in \mathcal{M}_{8\times 8}(\mathbb{C}) \mid A = -A^\top\}$, i.e., the skew-symmetric 8×8 matrices. The adjoint action on $\mathfrak{so}(8)$ is defined by:

$$\text{ad}_A : \begin{cases} \mathfrak{so}(8) & \to & \mathfrak{so}(8) \\ B & \mapsto & \text{ad}_A(B) = [A, B] = AB - BA \end{cases}$$

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 = -\lambda_1, \lambda_6 = -\lambda_2, \lambda_7 = -\lambda_3, \lambda_8 = -\lambda_4$ are the eigenvalues of A, the eigenvalues of ad_A are $\pm\lambda_i \pm \lambda_j$, $1 \leq i < j \leq 4$. A natural notion of discriminant for a Lie algebra is to consider the product of the roots (i.e., the linear functionals that generalize the notion of eigenvalues). As proven by Tevelev, the discriminant of $\mathfrak{so}(8)$ is nothing but the equation of the dual of the adjoint orbit. In other words, $A \in X^*_{SO(8)}$ if and only if

$$\Delta_{SO(8)}(A) = \prod_{1 \leq i < j \leq 4} (\lambda_i \pm \lambda_j)^2 = 0.$$

Let us denote by $h_i(A)$ the sum of the $i \times i$ principal minors of A and let $Q_A(t) = t^8 + h_2(A)t^4 + h_4(A)t^2 + \mathrm{Pf}(A)^2$ be the characteristic polynomial of A (recall that the $i \times i$ minors of a skew-symmetric matrix are zero if i is odd, and $\det(A) = \mathrm{Pf}(A)^2$, where $\mathrm{Pf}(A)$ is the Pfaffian). The roots of Q_A are $\pm\lambda_i$, $i = 1, \dots, 4$. Consider now $P_A(x) = Q_A(\sqrt{x}) = x^4 + h_2(A)x^2 + h_4(A)x + \mathrm{Pf}(A)^2$.

Then the roots of P_A are now the λ_i^2, $i = 1, \dots, 4$, and thus,

$$\Delta(P_A) = \Delta(x^4 + h_2(A)x^2 + h_4(A)x + \mathrm{Pf}(A)^2) = \prod_{1 \leq i < j \leq 4} (\lambda_i \pm \lambda_j)^2 = \Delta_{SO(8)}(A).$$

The polynomial ring of invariants $\mathbb{C}[\mathfrak{so}(8)]^{SO(8)}$ is generated by $h_2, h_4, h_6, \mathrm{Pf}$, and their restrictions to $(\mathbb{C}^2)^{\otimes 4}$ can be expressed in terms of H, L, M, D_{xy}:

$$\begin{array}{ll} h_{2|(\mathbb{C}^2)^{\otimes 4}} = 2B, & h_{4|(\mathbb{C}^2)^{\otimes 4}} = H^2 + 2L + 4M, \\ h_{6|(\mathbb{C}^2)^{\otimes 4}} = 2HL + 4HM - 4D_{xy}, & \mathrm{Pf}_{|(\mathbb{C}^2)^{\otimes 4}} = L. \end{array}$$

Leading by Theorem 2.2 to the following (alternative) expression [27] of HDet_{2222}:

$$\mathrm{HDet}_{2222} = \Delta\left(x^4 + 2Hx^3 + (H^2 + 2L + 4M)x^2 + (2HL + 4HM - 4D_{xy})x + L^2\right).$$

With Luke Oeding, we used Theorems 2.1 and 2.2 to deduce new explicit expressions of equations of dual varieties from the discriminant of the E_8 Lie algebra, exploiting several "branchings" similar to Eq. (2.9) to first establish division relations (as summerized in Figure 5). In particular, we obtained [28] an expression of the analogue of HDet_{2222} for four fermions with eight single-particle states, i.e., the dual equation of the Grassmannian $G(4,8) \subset \mathbb{P}(\wedge^4\mathbb{C}^8)$. In [29], we derived from the branching of $\mathfrak{e}_8$ with the spin module of $\mathfrak{so}(16)$ a hyperdeterminant for the $N = 8$ fermionic Fock space (four particle locations and two modes).

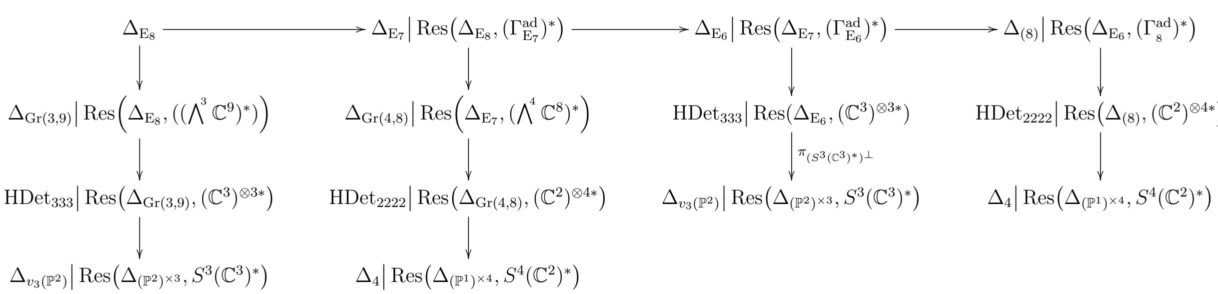

FIGURE 5. Division relations for a sequence of discriminants starting from the E_8-discriminant.

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3. Evaluating Algebraic Invariants with a Quantum Computer

We have shown that algebraic invariants, which can be computed using classical or geometric techniques, are meaningful in the context of quantum information processing. In [23, 30], we studied entanglement evolution in well-known quantum algorithms (Grover's and Shor's algorithms [38]) by examining the evolution of certain algebraic invariants during the iterations of the algorithms. However, what can physicists actually do with hyperdeterminants and other invariant polynomials at the experimental level? If these concepts are important for quantum information, how can they be applied to quantum device experiments?

3.1. **Crash Course on Quantum Algorithms.** Let us briefly introduce basic notions of quantum computing using the circuit formalism. An algorithm (classical or quantum) can be seen as a procedure defined by some input, a sequence of unambiguous finite operations, and an output. In classical computation, the input is represented by an m-bit string, the operations are a finite number of logical gates (universal set), and the output is an n-bit string.

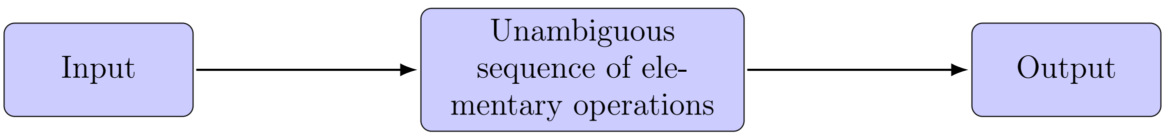

Figure 6. Simplest definition of what an algorithm is.

Thus, a classical algorithm can be viewed as the evaluation of a Boolean function $f : \mathbb{F}^m \to \mathbb{F}^n$ with $\mathbb{F} = \{0, 1\}$. Similarly, a quantum algorithm is a procedure that deals with an input given by an n-qubit quantum state, upon which a finite sequence of unitary transformations (quantum gates) is applied. The output of this process is a new quantum state, which must be measured (or partially measured) to obtain classical information.

Aspect	**Classical Algorithm**	**Quantum Algorithm**
Input	An m-bit string, $x = x_{m-1} \dots x_0$	A quantum state in some Hilbert space $\lvert\psi\rangle = \sum_{i_1,\dots,i_n \in \{0,1\}} a_{i_1,\dots,i_n} \lvert i_1 \dots i_n\rangle$
Operations	Sequential, deterministic steps on classical bits (logical gates: AND, OR, NOT) that encode a Boolean function f	Sequence of unitary operations (quantum gates, examples: X, Y, Z (Pauli matrices), CNOT gate) that encode a unitary operator U
Output	Definite output after deterministic processing $y = y_{m-1} \dots y_0 = f(x)$	Probabilistic output upon measurement of $\lvert\psi'\rangle = U \lvert\psi\rangle$

Table 6. Comparison of Classical and Quantum Algorithms.

The circuit formalism is a way of representing quantum algorithms or quantum computations. We work over an n-qubit Hilbert space, where each wire represents one qubit (a copy of the Hilbert space $\mathbb{C}^2$), and the gates represent unitary matrices acting on one or more qubits. Figure 7 shows an example of the action of the operator

$$(X \otimes U).(CNOT_{01} \otimes I_4).(I_2 \otimes CNOT_{12} \otimes I_2).(H \otimes X \otimes Z \otimes H)$$

on a 4-qubit input $|\psi\rangle = |q_0\rangle \otimes |q_1\rangle \otimes |q_2\rangle \otimes |q_3\rangle$, where

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{and} \quad CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

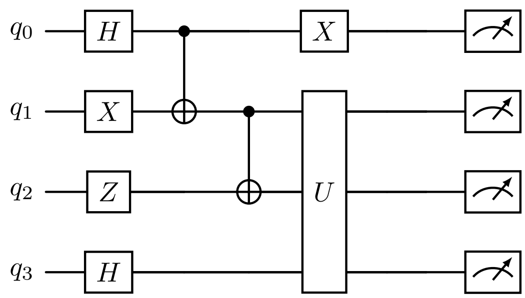

FIGURE 7. Example of a quantum algorithm in the circuit formalism. The quantum gates H, X, and Z are the Hadamard gate and the Pauli gates in the x and z directions. The CNOT gate is a controlled gate where the control qubit is represented by the black dot, and the target qubit is denoted by $\oplus$ (i.e., an X gate). The X gate is applied to the target if the control qubit is $|1\rangle$. The U gate is a unitary three-qubit gate acting on qubits 1, 2, and 3. At the end of the algorithm, the state $|\psi'\rangle = (X \otimes U).(CNOT_{01} \otimes I_4).(I_2 \otimes CNOT_{12} \otimes I_2).(H \otimes X \otimes Z \otimes H)\,|\psi\rangle$ is measured in the standard basis.

Nowadays, Noisy Intermediate Scale Quantum computers are available through the cloud[1] and quantum softwares, such as Qiskit[2] of MPQP[3], allows one to easily implement gate based computations.

3.2. **Measuring Non-Locality.** Entanglement is closely linked to the non-local properties of quantum physics, and the new era of quantum technologies is based on these counter-intuitive properties (superposition, entanglement, contextuality). Interestingly, the rise of these technologies originated from a philosophical debate between Einstein and Bohr about the nature of reality, famously known as the EPR paradox, after the paper by Einstein, Podolsky, and Rosen [12]. Roughly speaking, because the quantum correlations shared by entangled states violated the principle of locality, Einstein and his co-authors argued that quantum mechanics was incomplete and proposed the idea of hidden variable theories.

In the 1960s, John Bell transformed this philosophical question into a scientific one by proving that measurements of entangled states behave differently depending on whether quantum mechanics or local hidden variable theories hold true. This result is encapsulated in what is known as Bell's inequalities. Finally, the experiments by Aspect in the 1980s confirmed the correctness of quantum mechanics, paving the way for utilizing these once-philosophical debates as resources for new technologies.

[1] `https://quantum.ibm.com/`
[2] `https://github.com/qiskit`
[3] `https://mpqpdoc.colibri-quantum.com/`

Next, I will explain how Bell's inequalities can be tested on a quantum computer, which is a standard exercise for students learning quantum computing. Consider the following Bell observable:

$$\mathcal{B} = Z \otimes \left(\frac{Z+X}{\sqrt{2}}\right) + X \otimes \left(\frac{Z+X}{\sqrt{2}}\right) + Z \otimes \left(\frac{Z-X}{\sqrt{2}}\right) - X \otimes \left(\frac{Z-X}{\sqrt{2}}\right). \tag{3.1}$$

In quantum physics, an observable is a Hermitian operator that encodes the outcomes of a measurement. The eigenvectors of the observable define the measurement basis of the experiment, and the eigenvalues determine the possible measurement outcomes. Equation (3.1) represents a linear combination of four different experiments. The first qubit is measured in either the X or Z basis (the eigenbasis of X and Z), and the second qubit is measured in either the $\frac{Z+X}{\sqrt{2}}$ or $\frac{Z-X}{\sqrt{2}}$ basis. Since Pauli matrices have eigenvalues ± 1, the outcomes of the four measurements will be ± 1.

Assuming the existence of a hidden variable theory that respects the principles of Local Realism (LR)—i.e., the outcomes of a measurement pre-exist the measurement and are not influenced by distant measurements—one can show that the average value, $\langle \mathcal{B} \rangle^{LR}$, of the four experiments defined by the Bell operator is bounded by 2. However, under the rules of quantum mechanics (QM), the expectation value of $\mathcal{B}$ for a given two-qubit state $|\psi\rangle$ is given by

$$\langle \mathcal{B} \rangle_\psi = \langle \psi | \mathcal{B} | \psi \rangle \leq 2\sqrt{2}.$$

Equality is achieved for $|\psi\rangle = |EPR\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

The violation of Bell's inequality can be tested on a quantum computer by generating four circuits corresponding to the four terms of $\mathcal{B}$:

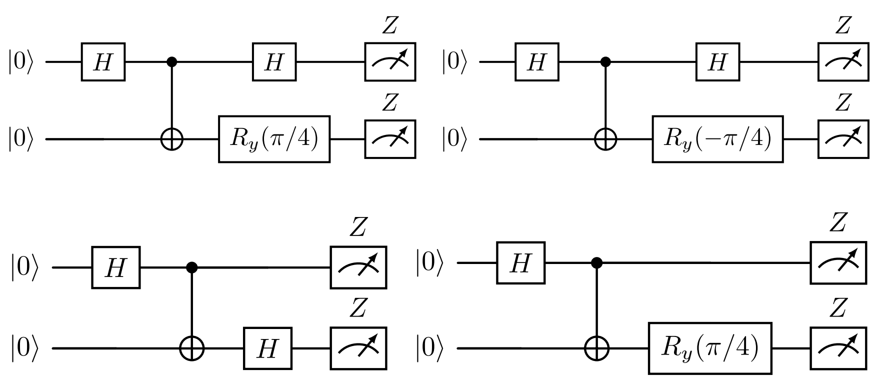

FIGURE 8. The four circuits represent the evaluation of the four terms $Z \otimes \left(\frac{Z+X}{\sqrt{2}}\right)$, $Z \otimes \left(\frac{Z-X}{\sqrt{2}}\right)$, $X \otimes \left(\frac{Z+X}{\sqrt{2}}\right)$, and $X \otimes \left(\frac{Z-X}{\sqrt{2}}\right)$ on the state $|EPR\rangle$. The Hadamard (H) and rotation gates before the measurement in the Z basis indicate a change of basis from the standard Z basis to the X and $\frac{Z\pm X}{\sqrt{2}}$ bases, respectively.

Bell's inequality has been generalized to n-qubit systems by Mermin [35], and it can be used to detect non-locality (and thus entanglement) by showing violations of the classical bound. By considering optimized Mermin operators, we can measure non-locality and quantify entanglement in quantum algorithms. In [2, 9], we studied the violation of the generalized

Bell's inequality in certain quantum algorithms (Grover's and Shor's algorithms, as well as the Quantum Fourier Transform) and for measuring the non-local behavior of hypergraph states. We compared this non-local invariant to $|\mathrm{HDet}_{2222}|$. In those papers, we pointed out similarities in the evolution of curves provided by the physical and algebraic invariants. A similar idea was proposed in [13], where the authors compared optimized Bell-like/Mermin operators with four-qubit invariant polynomials. They noted similarities between the curves of these operators and the algebraic invariant S for certain parameterized entangled states (see Figure 9).

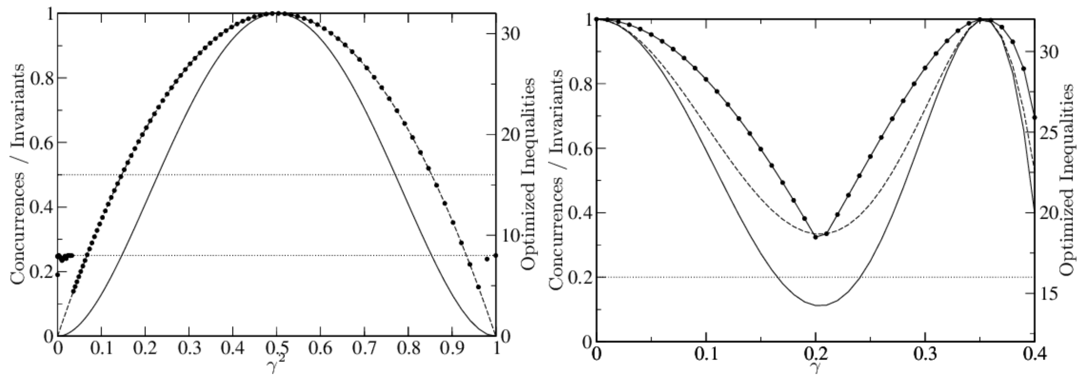

FIGURE 9. Numerical comparison of optimized Bell/Mermin inequalities (dots) with evaluations of the S invariant polynomial on two four-qubit entangled states, depending on a parameter γ. The dashed line represents another measure of entanglement (see [13]).

One of the main advantages of Bell/Mermin operators is that they can be directly measured on a quantum device [1]. If algebraic invariants are also meaningful for detecting and measuring entanglement, can we evaluate them on a quantum device?

3.3. **Measuring Algebraic Invariants.** Suppose we have a quantum state $|\psi\rangle = U|0\dots 0\rangle$ given by a "black box" unitary U and an invariant polynomial P that we would like to evaluate on $[\psi]$. Assuming $|\psi\rangle = \sum_{i_1,\dots,i_n\in\{0,1\}} a_{i_1,\dots,i_n}|i_1\dots i_n\rangle$, measurement of $|\psi\rangle$ only gives us access to $|a_{i_1,\dots,i_n}|^2$, which in most cases is not enough to evaluate P.

In [40], an interesting classical-quantum algorithm was proposed to evaluate $|\mathrm{HDet}_{222}(\psi)|$ for any three-qubit state $|\psi\rangle$. Recall that for $|\psi\rangle = a_{000}|000\rangle + a_{001}|001\rangle + \cdots + a_{111}|111\rangle$, one has:

$$\begin{aligned}\mathrm{HDet}_{222}(\psi) = {}& a_{000}^2a_{111}^2 + a_{010}^2a_{101}^2 + a_{001}^2a_{110}^2 + a_{011}^2a_{100}^2\\ &+ 4(a_{000}a_{011}a_{101}a_{110} + a_{001}a_{010}a_{100}a_{111})\\ &- 2(a_{000}a_{001}a_{110}a_{111} + a_{000}a_{010}a_{101}a_{111} + a_{000}a_{011}a_{100}a_{111}\\ &\quad + a_{001}a_{010}a_{101}a_{110} + a_{001}a_{011}a_{100}a_{110} + a_{010}a_{011}a_{100}a_{101}).\end{aligned}$$

The group of local unitary operations $LU = U_2(\mathbb{C})\times U_2(\mathbb{C})\times U_2(\mathbb{C})$ has real dimension 9, while the number of real parameters to describe $|\psi\rangle$ is $14 = 16-2$ (the real dimension of $\mathcal{H}_{222}$ minus the normalization constraint and the global phase equivalence). Therefore, one can show that it is always possible, up to local unitary transformation, to find six real amplitudes $\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4$, and φ such that:

$$|\psi\rangle \equiv_{LU} \alpha_0|000\rangle + \alpha_1 e^{i\varphi}|100\rangle + \alpha_2|101\rangle + \alpha_3|110\rangle + \alpha_4|111\rangle\,. \qquad (3.2)$$

But HDet is invariant under the local unitary group, and therefore

$$|\mathrm{HDet}(\psi)| = \alpha_0^2\alpha_4^2.$$

Thus, if $|\psi\rangle$ is transformed into its canonical form (Eq. (3.2)), the hyperdeterminant evaluated at $[\psi]$ is nothing but the product of the frequency of measuring $|000\rangle$ and the frequency of measuring $|111\rangle$.

In order to find the parameters of the one-qubit unitary gates $U_1, U_2, U_3 \in U_2(\mathbb{C})$ that transform $|\psi\rangle$ into its canonical form, we consider the following Variational Quantum Circuit (VQC):

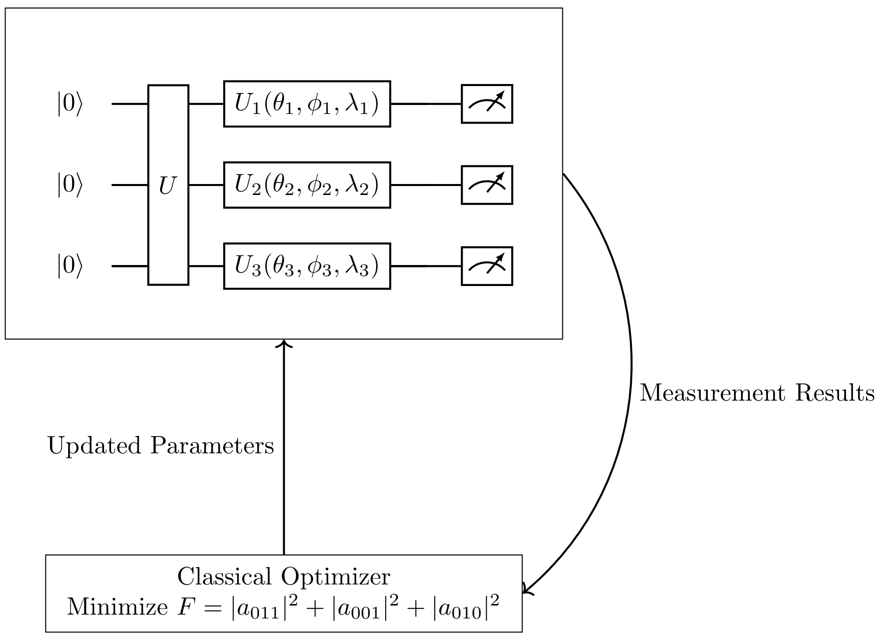

FIGURE 10. Variational quantum algorithm to compute $\mathrm{HDet}([\psi])$ with $|\psi\rangle = U|000\rangle$ for an unknown U.

In [40], the authors simulate their algorithm under noisy environments and propose error mitigation techniques. Their algorithm allows one to recognize if a three-qubit state is SLOCC equivalent to $|GHZ\rangle$ or not.

Let's extend their idea to provide a quantum algorithm that, given a quantum state $|\psi\rangle = U|000\rangle$, allows one to find its class of entanglement. Recall from Section 1 that if $\mathrm{HDet}_{222}([\psi]) = 0$, it is possible to decide if $|\psi\rangle$ is SLOCC equivalent to $|W\rangle$ by evaluating the rank of the 2×4 flattening of $|\psi\rangle$. In the canonical form obtained by the VQC one has

$$\varphi^1 = \begin{pmatrix} a_{000} & 0 & 0 & 0 \\ a_{100} & a_{101} & a_{110} & a_{111} \end{pmatrix}.$$

Therefore, the rank of φ^1 can be obtained from the frequencies of the measurements on $|000\rangle, |100\rangle, |101\rangle, |110\rangle$, and $|111\rangle$, which will give approximations of $|a_{000}|^2$, $|a_{100}|^2$, $|a_{101}|^2$, $|a_{110}|^2$, and $|a_{111}|^2$. The ranks of φ^2 and φ^3 cannot be obtained from the canonical form given by Eq. (3.2), but alternative canonical forms can be obtained by changing the loss function.

Remark 4. As noted in [40], the optimization process requires repeating the experiment several times and is not better in terms of time complexity than quantum tomography, which consists of recovering the amplitudes of $|\psi\rangle$ from measurements in different bases.

Can we apply similar VQC techniques in the four-qubit case? Let $|\psi\rangle$ be a four-qubit state defined by $30 = 32 - 2$ real coefficients. The group $LU = U_2(\mathbb{C})^{\times 4}$ has real dimension 12, which means that one could, in the best-case scenario, set 6 complex amplitudes (or 12 real parameters) to zero out of 16. For instance, it would not be possible to deduce from the amplitudes the rank of the 2×8 flattenings. Similarly, to obtain the value of L or M from the frequencies of the measurements, one needs first to transform the state into a form with 6 amplitudes set to zero. However, the rank of the 2×8 flattening and the determinant of the 4×4 flattening are respectively $U_2(\mathbb{C}) \times U_8(\mathbb{C})$ and $U_4(\mathbb{C}) \times U_4(\mathbb{C})$ invariants. This allows us to use a larger space of transformations to obtain canonical forms that are not LU equivalent but for which the invariants we consider are the same.

Problem 1. *Find a quantum algorithm that, given a 4-qubit state provided by a black box U, $|\psi\rangle = U|0000\rangle$, determines the tensor rank of $|\psi\rangle$.*

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