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Anticoncentration of Random Matrix Product States

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"Anticoncentration of random tensor network states"
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Anticoncentration and state design of random tensor networks

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We investigate quantum random tensor network states where the bond dimensions scale polynomially with the system size, N. Specifically, we examine the delocalization properties of random Matrix Product States (RMPS) in the computational basis by deriving an exact analytical expression for the Inverse Participation Ratio (IPR) of any degree, applicable to both open and closed boundary conditions. For bond dimensions $\chi \sim \gamma N$, we determine the leading order of the associated overlaps probability distribution and demonstrate its convergence to the Porter-Thomas distribution, characteristic of Haar-random states, as γ increases. Additionally, we provide numerical evidence for the frame potential, measuring the 2-distance from the Haar ensemble, which confirms the convergence of random MPS to Haar-like behavior for $\chi \gg \sqrt{N}$. We extend this analysis to two-dimensional systems using random Projected Entangled Pair States (PEPS), where we similarly observe the convergence of IPRs to their Haar values for $\chi \gg \sqrt{N}$. These findings demonstrate that random tensor networks with bond dimensions scaling polynomially in the system size are fully Haar-anticoncentrated and approximate unitary designs, regardless of the spatial dimension.

Matrix Product States (MPS)

$$\begin{cases} \mathcal{H} \sim \bigotimes_{i=1}^{N} \mathbb{C}^d \\ d = \text{local qudit dimension} \\ N = \text{number of qudits} \\ D = d^N = \text{dimension of } \mathcal{H} \end{cases}$$

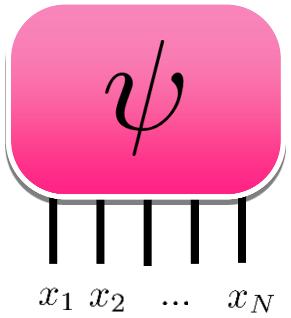
Computational basis:

$|x_i\rangle \in \{0, 1 \dots d-1\}$

$|\boldsymbol{x}\rangle = |x_1, x_2 \dots x_N\rangle$

Arbitrary state vector $|\psi\rangle \in \mathcal{H}$:

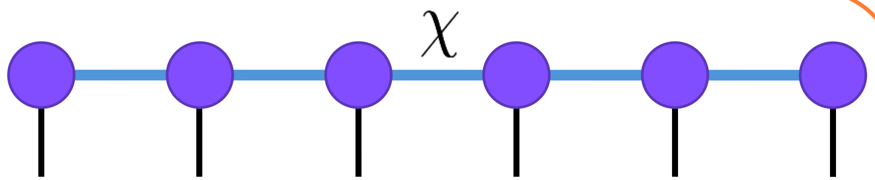
$$|\psi\rangle = \sum_{\boldsymbol{x}} \psi_{\boldsymbol{x}} |\boldsymbol{x}\rangle$$



The many-body problem: tensor ψ is exponentially complex

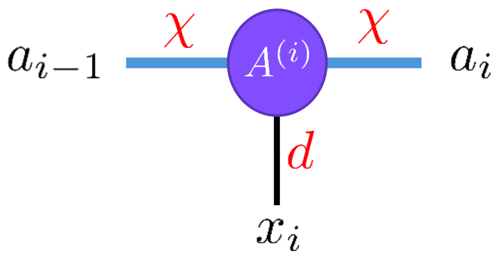
MPS

$$|\psi\rangle = \sum_{\boldsymbol{x}} \sum_{a_1, a_2 \dots a_{N-1}} A^{(1)}_{1 a_1}(x_1) A^{(2)}_{a_1 a_2}(x_2) \dots A^{(N)}_{a_{N-1} 1}(x_N) |\boldsymbol{x}\rangle$$



$a_i \in \{1, 2, \dots \chi\}$

χ= bond dimension $\rightarrow$ Controls the level of correlations / entanglement

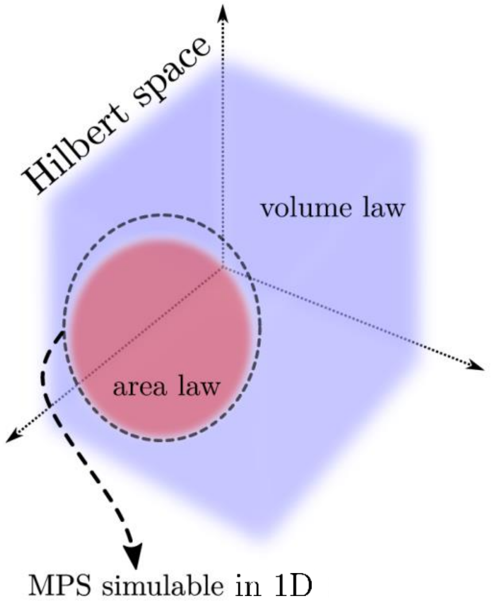


Matrix Product States (MPS)

MPS are prototypical many-body states!

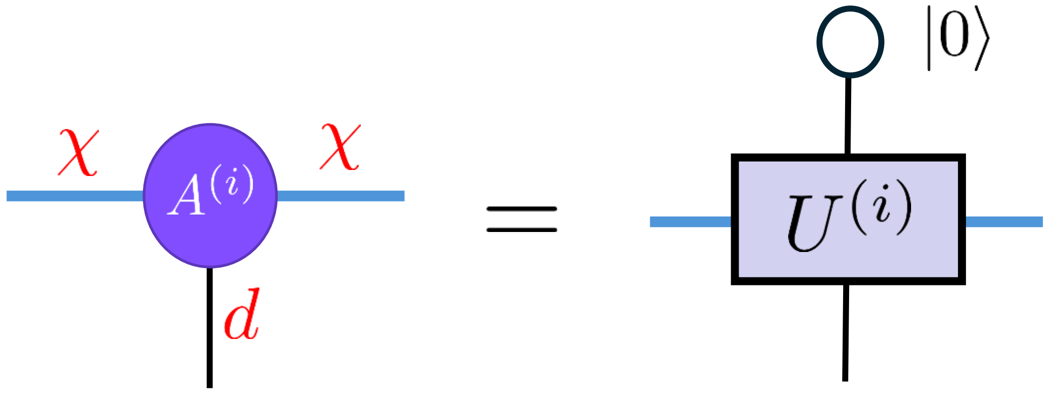
- ground-states in 1D are MPS with finite bond dimension
- relatively easy to generate in lab (digital quantum platforms)
- extremly useful in numerical simulations (DMRG, TEBD, TDVP, etc.)

However many *statistical properties* are still not well explored!



Random Matrix Product States (RMPS)

MPS tensors are sub-blocks of Haar matrices:



$U^{(i)} \sim$ unitary Haar matrix of size $d\chi$

General questions:

- How ergodic is this ensemble?
- How well does it approximate Haar states when scaling χ with N?

S. Garnerone, T. R. de Oliveira, P. Zanardi (2009)

J. Haferkamp, C. Bertoni, I. Roth, J. Eisert (2021)

C. Lancien, D. Perez-García (2019)

Inverse Participation Ratio (IPR)

Quantifying the anticoncentration of a state over the computational basis:

$$\mathcal{I}^{(k)}(|\psi\rangle) \equiv \sum_{\boldsymbol{x}} |\langle \boldsymbol{x}|\psi\rangle|^{2k} \qquad k \in \mathbb{N}$$

$\mathcal{I}^{(1)}(|\psi\rangle) = 1$ for normalized states $\langle\psi|\psi\rangle = 1$

$\mathcal{E} = \{|\psi^{(\mu)}\rangle\}_{\mu=1}^{K} \rightarrow$ ensemble of states $\qquad \mathcal{I}^{(k)}_{\mathcal{E}} \equiv \mathbb{E}_{\psi\sim\mathcal{E}}[\mathcal{I}^{(k)}(|\psi\rangle)]$

$\mathcal{P}(w) = \mathbb{E}_{\boldsymbol{x}\sim\mathcal{B},\psi\sim\mathcal{E}}\left[\delta\left(w - D|\langle \boldsymbol{x}|\psi\rangle|^2\right)\right] =$ distribution of overlaps $|\langle \boldsymbol{x}|\psi\rangle|^2$

k-th IPR $\propto$ k-th moment of $\mathcal{P}$

Porter-Thomas distribution:

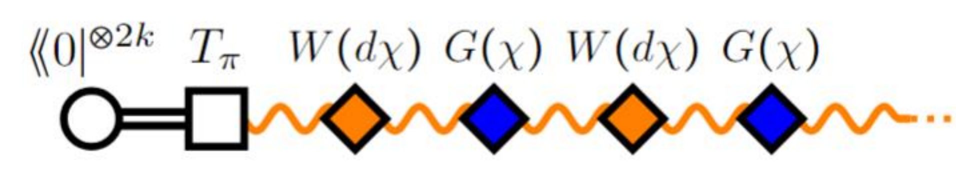
$\mathcal{P}(w) = e^{-w}$, for $|\psi\rangle = U|0...0\rangle$, $U \sim \mathrm{Haar(D)}$

Inverse Participation Ratio (IPR) of RMPS

Our results

1. Exact formula for for IPRs ($r = \log_d \chi$)

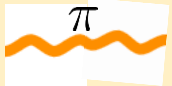
$$\mathcal{I}^{(k)}_{RMPS} = D\left(\frac{(\chi+1)\dots(\chi+k-1)}{d(d\chi+1)\dots(d\chi+k-1)}\right)^{N-r-1}\frac{k!}{d\chi(d\chi+1)\dots(d\chi+k-1)}$$



$|T_\pi\rangle\rangle$=permutation operator

$G_{\pi\sigma} = \langle\langle T_\pi | T_\sigma\rangle\rangle$

permutation index $\pi \in \{1, 2, \dots k!\}$

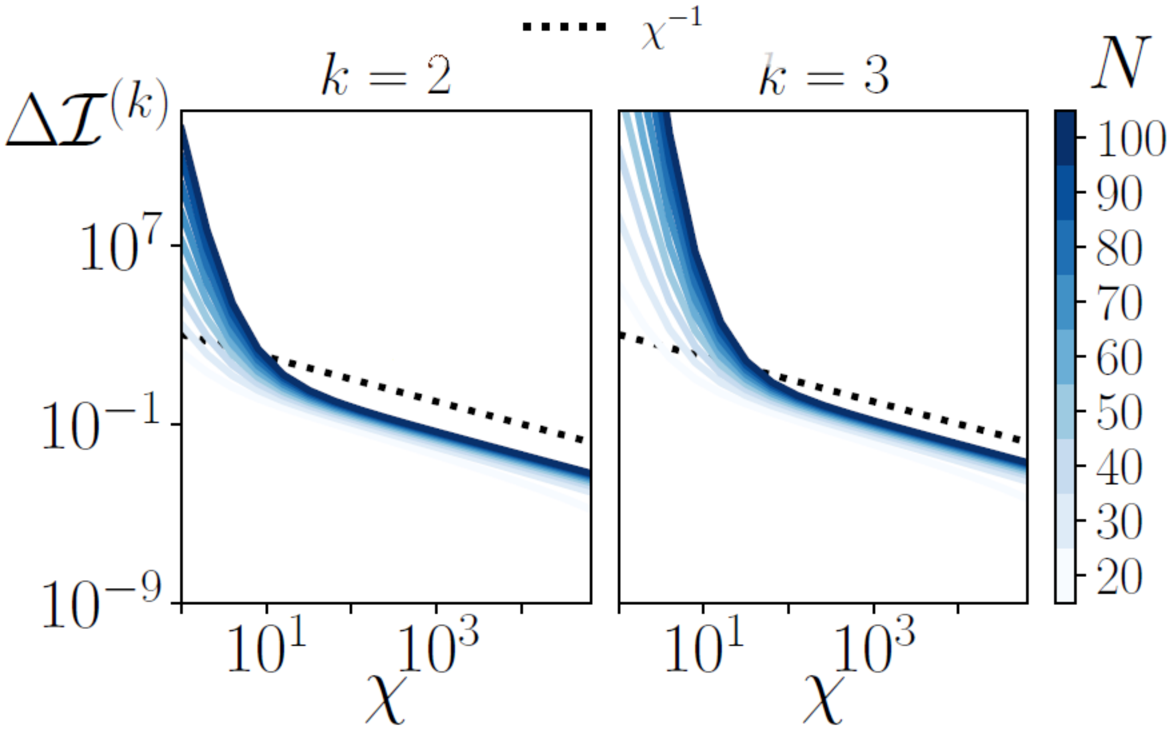


2. Distribution $\mathcal{P}(w)$ in the scaling limit $N, \chi \to \infty$ with constant χ/N

$$\lim_{\substack{N\to\infty \\ \chi = N\gamma(d-1)/d}} \mathcal{I}^{(k)}_{RMPS} = \mathcal{I}^{(k)}_{Haar} e^{k(k-1)/(2\gamma)}$$

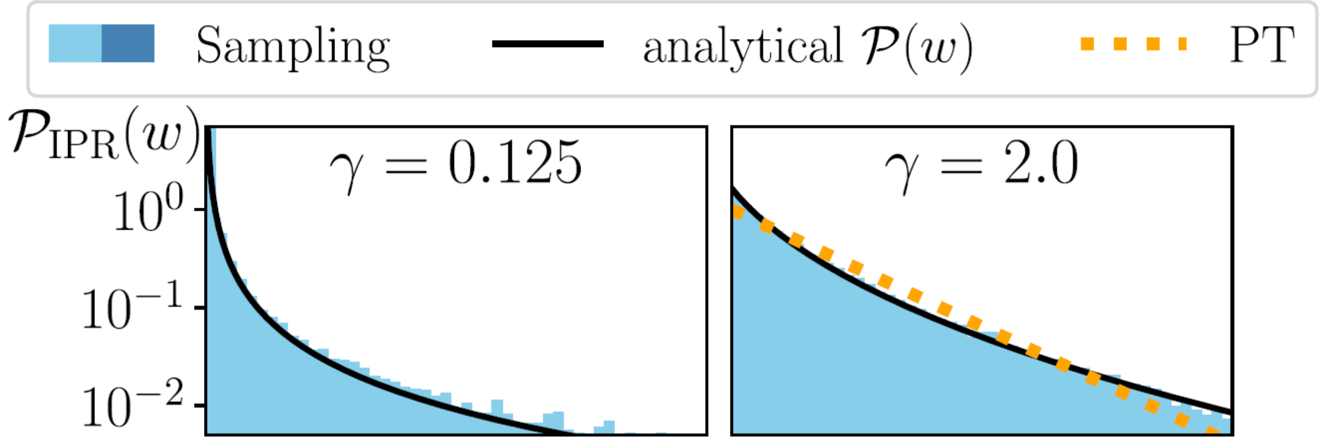
$$\mathcal{P}_{RMPS}(w;\gamma) = \int \frac{du}{\sqrt{2\pi}} e^{-u^2/2+1/\gamma} \exp\left(-we^{u/\sqrt{\gamma}+\frac{3}{2\gamma}}\right)$$

Inverse Participation Ratio (IPR) of RMPS



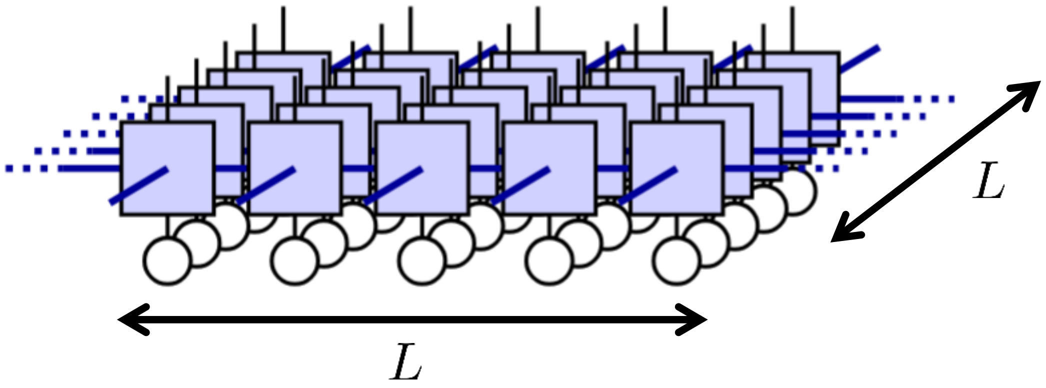
$$\Delta\mathcal{I}^{(k)} = \mathcal{I}^{(k)}_{RMPS}/\mathcal{I}^{(k)}_{Haar} - 1$$

corrections to Haar goes down with χ/N



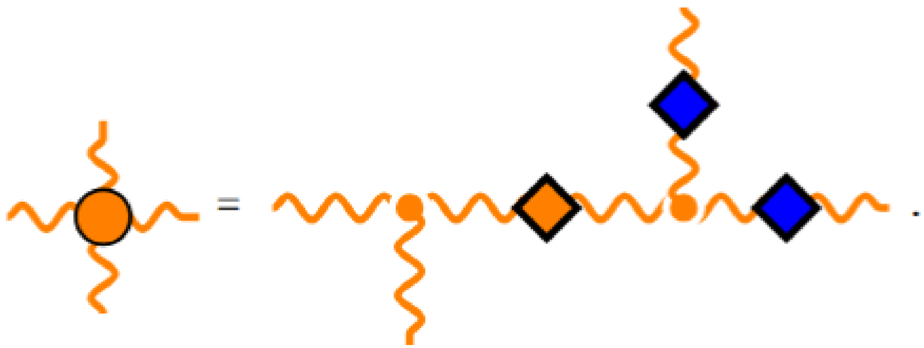
prediction for $\mathcal{P}(w)$ works also for small γ, χ

Inverse Participation Ratio (IPR) of Random PEPS

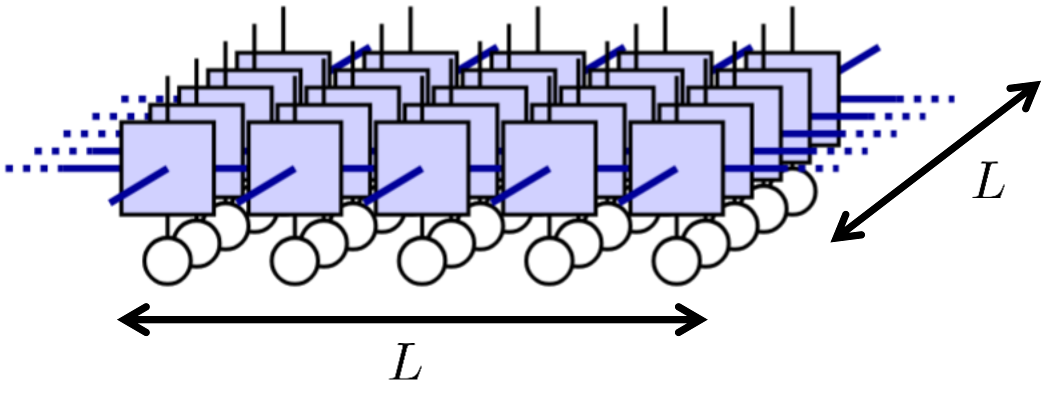


Random PEPS with bond dimension χ

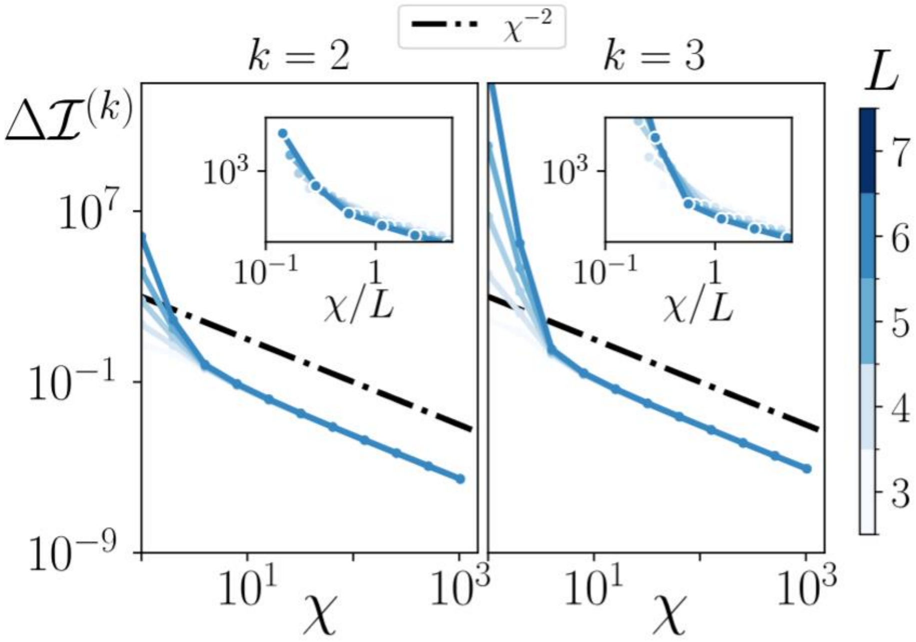
Numerical results by Tensor Network contraction
in the replicas (permutation) space



Inverse Participation Ratio (IPR) of Random PEPS



Random PEPS with bond dimension χ



$$\Delta\mathcal{I}^{(k)} = \mathcal{I}^{(k)}_{RMPS}/\mathcal{I}^{(k)}_{Haar} - 1$$

crossover at $\chi \sim L$, after which $\Delta\mathcal{I}^{(k)}$ goes down as χ^{-2}

Thank you!