Counting of $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

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Random Tensors

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Enumeration of $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

with Rémi Cocou Avohou, Joseph Ben Geloun

Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404 [hep-th]]

Summary of the main results

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

We enumerated $\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants using group theoretic formulas.

- Our enumerations unveiled a wide array of novel integer sequences that have not been previously known.
- For a general order (r, q), the counting can be interpreted as the partition function of a topological quantum field theory (TQFT) with the symmetric group as the gauge group. We identified the 2-complex pertaining to the enumeration of the invariants, which in turn defines the TQFT, and establish a correspondence with countings associated with covers of diverse topologies, in general with branched points.
- At order $(r, q) = (1, 1)$, the numbers of invariants corresponds to the numbers of certain cicular words with pattern avoidance, offering insights into enumerative combinatorics and potentially to linguistics.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

Consider

- A tensor T transforms under the action of the fundamental representation of the Lie group $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$.

$$T_{a_1,a_2,\ldots,a_r,b_1,b_2,\ldots,b_q} \to U^{(1)}_{a_1 c_1} U^{(2)}_{a_2 c_2} \ldots U^{(r)}_{a_r c_r} O^{(1)}_{b_1 d_1} O^{(2)}_{b_2 d_2} \ldots O^{(q)}_{b_q d_q} T_{c_1,c_2,\ldots,c_r,d_1,d_2,\ldots,d_q} \, .$$

- A $(\bigotimes_{i=1}^{r} \mathrm{U}(N_i)) \otimes (\bigotimes_{j=1}^{q} \mathrm{O}(D_j))$ invariant (UO-invariant) is constructed by contractions of complex tensors of order $r+q$ (a given number, n, of tensors T and the same number of complex conjugate $\bar{T}$.)

  → Therefore, UO invariants are tensor model invariants/bubbles.

- An UO-invariant is algebraically denoted

$$\mathrm{Tr}_{K_n}(T,\bar{T}) = \sum_{a_k^i, b_k^i, a_k'^i, b_k'^i} K_n(\{a_k^i, b_k^i\}; \{a_k'^i, b_k'^i\}) \prod_{i=1}^{n} T_{a_1^i,a_2^i,\ldots,a_r^i,b_1^i,b_2^i,\ldots,b_q^i} \bar{T}_{a_1'^i,a_2'^i,\ldots,a_r'^i,b_1'^i,b_2'^i,\ldots,b_q'^i} \, .$$

  K_n is a kernel composed of a product of Kronecker delta functions that match the indices of n copies of T's and those of n copies of $\bar{T}$'s.
  A given tensor contraction dictates the pattern of an edge-colored graph, which can, in turn, be used to label the tensor invariant.

$\mathrm{U}(N)^{\otimes r} \otimes \mathrm{O}(D)^{\otimes q}$ tensor invariants

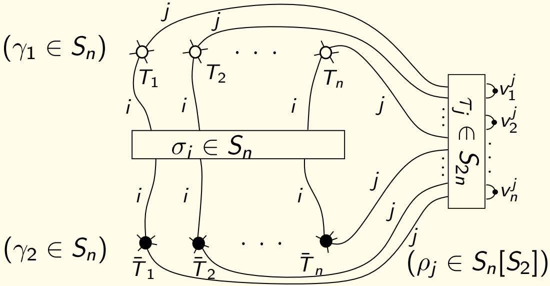


Diagram of contraction of n tensors T and n tensors $\bar{T}$. For a given color $i = 1, 2, \ldots, r$, σ_i represents the contraction in the unitary sector and, for any color $j = 1, 2, \ldots, q$, τ_j represents the contraction in the orthogonal sector.

Consider $(r, q) = (3, 3)$. An UO-invariant is defined by a $(3+3)$-tuple of permutations $(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3)$ from the product space $(S_n)^{\times 3} \times (S_{2n})^{\times 3}$.

We will remove the vertex labeling (two configurations are equivalent if their resulting unlabeled graphs coincide), which introduces more permutations $\gamma_1, \gamma_2 \in S_n$, and $\varrho_1, \varrho_2, \varrho_3 \in S_n[S_2]$ the so-called wreath product subgroup of S_{2n}.

The equivalence relation is

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \quad \sim \quad (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3)$$

Counting UO tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

(Idea)

We work with the equivalence relation to count the graphs, i.e., tensor invariants

$$(\sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \tau_3) \quad \sim \quad (\gamma_1\sigma_1\gamma_2,\ \gamma_1\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2,\ \gamma_1\gamma_2\tau_1\varrho_1,\ \gamma_1\gamma_2\tau_2\varrho_2,\ \gamma_1\gamma_2\tau_3\varrho_3)\,.$$

- $G \times X \to X$.
- Recall: orbit of an element x in X: the set of elements in X to which x can be moved by the elements of G. $G \cdot x = \{g \cdot x : g \in G\}$.
- a point ($\in X$) on an orbit $\to$ another point on the orbit.
- number of equivalent classes of graphs = number of orbits
- Burnside's lemma
  $\sharp\mathrm{orb} = \frac{1}{|G|}\sum_{g\in G} |\mathrm{Fix}(g)|$, where $\mathrm{Fix}(g) = \{x \in X : gx = x\}$.

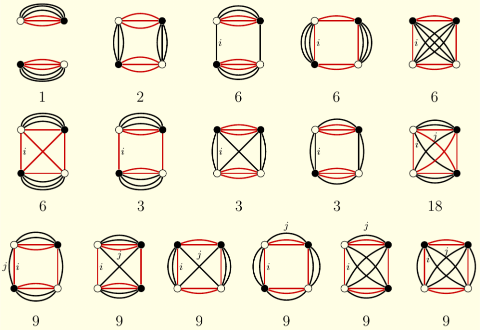
Therefore the counting of UO invariants of order (r, q) is

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q} \sum_{\gamma_1,\gamma_2\in S_n} \sum_{\varrho_1,\dots,\varrho_q\in S_n[S_2]} \sum_{\substack{\sigma_1,\dots,\sigma_r\in S_n \\ \tau_1,\dots,\tau_q\in S_{2n}}} \left[\prod_{i=1}^{r} \delta(\gamma_1\sigma_i\gamma_2\sigma_i^{-1})\right]\left[\prod_{i=1}^{q} \delta(\gamma_1\gamma_2\tau_i\varrho_i\tau_i^{-1})\right].$$

example: $\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

$\mathrm{U}(N)^{\otimes 3} \otimes \mathrm{O}(D)^{\otimes 3}$ tensor invariants are enumerated in the increasing number of tensors: $1, 108, 20385, 27911497, 101270263373, 808737763302769, \ldots$



UO-invariant graphs at order $(r, q) = (3, 3)$ with 4 tensors ($n = 2$). The integer below each graph enumerates various possibilities based on index colors, summing to 108 for all configurations. Black edges are in the U-sector, and red are in the O-sector.

TQFT (lattice gauge theories)

- On a **cellular complex** X, we can define a partition function for a finite **group** G by assigning a group element g_e to each edge (1-cell) and to each **plaquette** (2-cell) P a weight $w_P\Big(\prod_{e\in P} g_e\Big)$. The partition function of this **lattice gauge theory** is

$$Z[X;G] = \frac{1}{|G|^V} \sum_{g_e} \prod_P w_P\Big(\prod_{e\in P} g_e\Big),$$

  with V the number of vertices in the cell decomposition.

- The theory is **topological** because it is invariant under refinement of the cellular decomposition.
- When $G = S_n$ (symmetric group or permutation group), it has applications to QFT combinatorics. [Ben Geloun, Ramgoolam, Ann. Inst. H. Poincare Comb. Phys. Interact. 1 (2014) 1]
- The partition function counts equivalence classes of homomorphisms from $\pi_1(X)$ to S_n, i.e., counts equivalence classes of covering spaces of X of degree n counted with a certain weight.

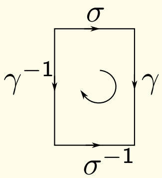
an example of permutation TQFT

e.g., Consider the torus realised as a rectangle.

- The partition function of this lattice gauge theory is given by

$$Z(T^2; S_n) = \frac{1}{n!} \sum_{\sigma,\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\sigma^{-1})\,.$$

- $\delta(\gamma\sigma\gamma^{-1}\sigma^{-1})$ or $\gamma\sigma\gamma^{-1}\sigma^{-1} = \mathrm{id}$ is represented by the torus and γ and σ are the generators of the fundamental group of the torus.



- $Z(T^2; S_n)$ counts n-fold covers of the torus.

permutation TQFT for UO tensor invariants

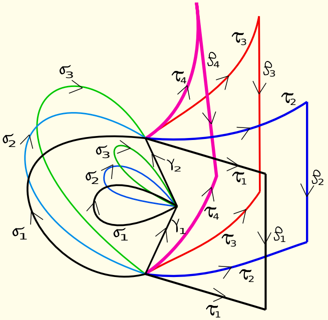
[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

Recall the counting of UO invariants of order (r, q)

$$Z_{(r,q)}(n) = \frac{1}{(n!)^2[n!(2!)^n]^q}$$

$$\sum_{\gamma_1,\gamma_2 \in S_n} \sum_{\varrho_1,\ldots,\varrho_q \in S_n[S_2]} \sum_{\substack{\sigma_1,\ldots,\sigma_r \in S_n \\ \tau_1,\ldots,\tau_q \in S_{2n}}} \left[\prod_{i=1}^{r} \delta(\gamma_1 \sigma_i \gamma_2 \sigma_i^{-1})\right]\left[\prod_{i=1}^{q} \delta(\gamma_1 \gamma_2 \tau_i \varrho_i \tau_i^{-1})\right].$$

TQFT reformulates our enumeration.



2-cellular complex associated with the $TQFT_2$ of $Z_{(3,4)}$ made of 3+4 cylinders sharing boundaries.

permutation TQFT for UO tensor invariants

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

The counting of UO invariants of order (r, q) can be massaged:

$$Z_{(r,q)}(n) = \frac{1}{n!} \sum_{\gamma \in S_n} Z^q_{n;\gamma} \sum_{\sigma_0, \sigma_2, \sigma_3 \ldots, \sigma_r \in S_n} \Big[\prod_{i=2}^{r} \delta(\gamma^{-1} \sigma_i \gamma \sigma_i^{-1}) \Big] \delta(\gamma^{-1} \sigma_0 \gamma \sigma_0^{-1}) \delta(\sigma_0 \prod_{i=2}^{r} \sigma_i) \,,$$

$$\text{with} \quad Z^q_{n;\gamma} = \frac{1}{(n!)[n!(2!)^n]^q} \sum_{\substack{\sigma_1 \in S_n \\ \tau_1, \ldots, \tau_q \in S_{2n} \\ \tau_i^{-1} \sigma_1 \gamma^{-1} \sigma_1^{-1} \gamma \tau_i \in S_n[S_2]}} 1 \,.$$

We are counting equivalence classes of r permutations σ_i under the conjugation $\sigma_i \sim \gamma \sigma_i \gamma^{-1}$, and the group generated by r generators subject to one relation by the last constraint $\sigma_0 \prod_{i=2}^{r} \sigma_i = \text{id}$, i.e., the fundamental group of the 2-sphere with r-punctures.

Therefore, $Z_{(r,q)}(n)$ counts the covers of the r-punctured sphere with each cover weighted by $Z^q_{n;\gamma}$, i.e., enumerates $Z^q_{n;\gamma}$-weighted equivalence classes of branched covers (with r branched points of degree n) of the sphere.

Consequences and Outlook

[Avohou, Ben Geloun, Toriumi, Eur. Phys. J. C 84, 839 (2024) [arXiv:2404.16404]]

- We added more correspondence between the enumeration of tensor invariants and 2-dimensional permutation TQFT.
- The sequences of numbers corresponding to our enumerations [1] [2] are new and unknown before in OEIS (Online Encyclopedia of Integer Sequences).
- So far, regardless of whether the invariants are unitary [Ben Geloun, Ramgoolam 2013], orthogonal [Avohou, Ben Geloun, Dub 2019], or UO symmetric, we consistently find a correspondence with (branched) covers of either the sphere or the torus (possibly with punctures). We ask what about non-orientable manifolds, e.g., the projective plane, the Klein bottle (as a closed manifold)? Which types of tensors, transformations, and tensor contractions may lead to the enumeration of covers of nonorientable manifolds?

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[1] except purely U case ($r, q = 0$) was reported before [Ben Geloun, Ramgoolam 2013] and also ($r = 2, q = 1$) case was reported in [Bulycheva, Klebanov, Milekhin Tarnopolsky 2017].

[2] Remark that our formulation cannot be reduced to purely O case which was studied before [Avohou, Ben Geloun,Dub 2022].

Take home messages

The counting of tensor invariants, in addition to their essential role in the pertubative analysis of tensor models in theoretical physics, reveals unexpected connections between combinatorics, algebra, and topology.

What is intriguing is the connection between tensor models and branched covers of the 2-sphere suggests that two dimensional holomorphic maps know about higher dimensional combinatorial topology.