Matrix Models for Matter on Random Geometries with Causal Constraints

Juan L. A. Abranches

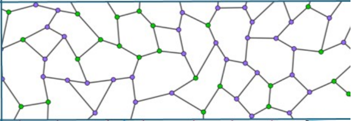
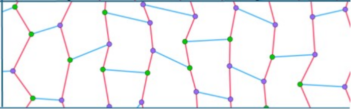
with Antonio D. Pereira and Reiko Toriumi

Okinawa Institute of Science and Technology - Okinawa, Japan

2024 October 2

Random tensors and related topics - IHP 2024

Main aspects

Matrix Models	Zero-dimensional QFT of random matrices. Ribbon graphs as Feynman graphs.	$Z = \int \prod_{i,j} dA_{ij}\, e^{-S[A]}$
Ribbon Graphs	Vertices and edges are topological discs. Disks intersect at disjoint line segments. Each segment borders one vertex and one edge. Each edge has two segments in its border.	
Matter Interaction	Ising model. Vertices represent matter. Edges represent matter interaction.	
2D Causal Structure	Causal dynamical triangulations. Only topologies with time foliation. No spatial topology change.	
2D Quantum Gravity	Ising model & causal dynamical triangulations. Interaction between matter and spacetime.	

Outline

- Implementing the causality constraint at the level of a matrix model is a first step towards its implementation in tensor models.

- We defined and studied a matrix model that describes the Ising Model coupled to the Causal Dynamical Triangulations (CDT).

- We revisited a problem that appeared on the CDT Matrix Model, which is finding the Gaussian average of the character of the square of Hermitian matrices.

- The inclusion of the Ising model produces a multi-matrix model which allows us to explore the impact of the dynamical (and causal) lattice to the Ising model and vice-versa.

Outline

- Implementing the causality constraint at the level of a matrix model is a first step towards its implementation in tensor models.
- We defined and studied a matrix model that describes the Ising Model coupled to the Causal Dynamical Triangulations (CDT).
- We revisited a problem that appeared on the CDT Matrix Model, which is finding the Gaussian average of the character of the square of Hermitian matrices.
- The inclusion of the Ising model produces a multi-matrix model which allows us to explore the impact of the dynamical (and causal) lattice to the Ising model and vice-versa.

Outline

- Implementing the causality constraint at the level of a matrix model is a first step towards its implementation in tensor models.
- We defined and studied a matrix model that describes the Ising Model coupled to the Causal Dynamical Triangulations (CDT).
- We revisited a problem that appeared on the CDT Matrix Model, which is finding the Gaussian average of the character of the square of Hermitian matrices.
- The inclusion of the Ising model produces a multi-matrix model which allows us to explore the impact of the dynamical (and causal) lattice to the Ising model and vice-versa.

Outline

- Implementing the causality constraint at the level of a matrix model is a first step towards its implementation in tensor models.
- We defined and studied a matrix model that describes the Ising Model coupled to the Causal Dynamical Triangulations (CDT).
- We revisited a problem that appeared on the CDT Matrix Model, which is finding the Gaussian average of the character of the square of Hermitian matrices.
- The inclusion of the Ising model produces a multi-matrix model which allows us to explore the impact of the dynamical (and causal) lattice to the Ising model and vice-versa.

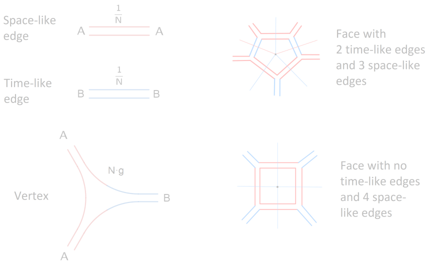
Matrix Model for the CDT Model [D. Benedetti, J. Henson, 2009]

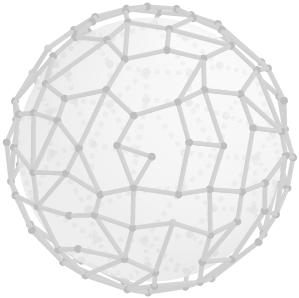
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

$$Z = \int dAdB\, e^{-S_{CDT}} , \qquad \text{where} \qquad S_{CDT} = N\,\mathrm{Tr}\left[\frac{1}{2}A^2 + \frac{1}{2}\left(C_2^{-1}B\right)^2 - gA^2B\right] .$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$.





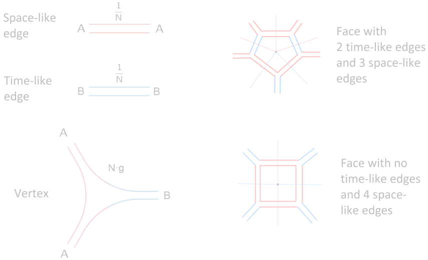
Matrix Model for the CDT Model [D. Benedetti, J. Henson, 2009]

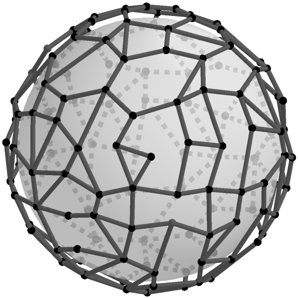
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

$$Z = \int dAdB \, e^{-S_{CDT}} \,, \qquad \text{where} \qquad S_{CDT} = N \, \mathrm{Tr} \left[\frac{1}{2} A^2 + \frac{1}{2} \left(C_2^{-1} B \right)^2 - g A^2 B \right] .$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N \delta_{p,2}$.





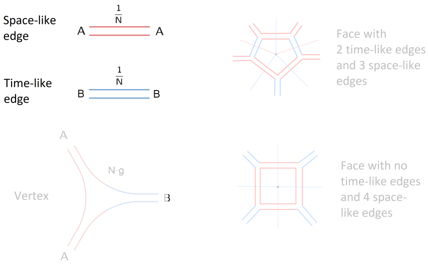
Matrix Model for the CDT Model [D. Benedetti, J. Henson, 2009]

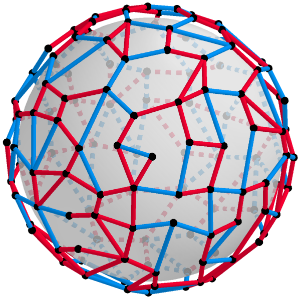
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

$$Z = \int dAdB\, e^{-S_{CDT}}\ , \qquad \text{where} \qquad S_{CDT} = N\,\mathrm{Tr}\left[\frac{1}{2}A^2 + \frac{1}{2}\left(C_2^{-1}B\right)^2 - gA^2B\right] .$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$.





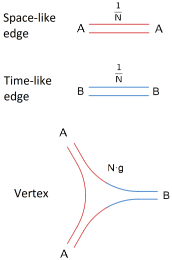
Matrix Model for the CDT Model [D. Benedetti, J. Henson, 2009]

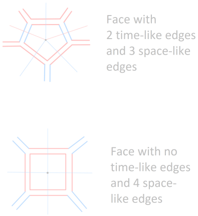
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

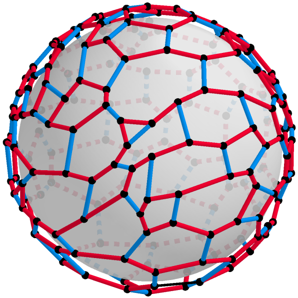
$$Z = \int dAdB\, e^{-S_{CDT}}\ , \qquad \text{where} \qquad S_{CDT} = N\,\mathrm{Tr}\left[\frac{1}{2}A^2 + \frac{1}{2}\left(C_2^{-1}B\right)^2 - gA^2B\right].$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$.





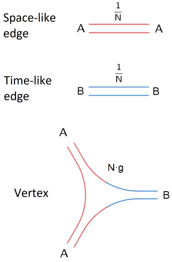


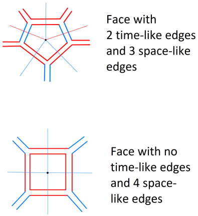
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

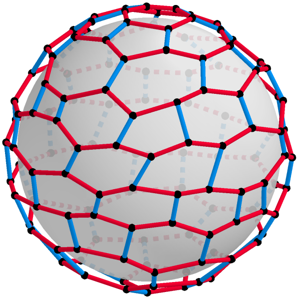
$$Z = \int dAdB \; e^{-S_{CDT}} \, , \qquad \text{where} \qquad S_{CDT} = N \, \mathrm{Tr} \left[\frac{1}{2} A^2 + \frac{1}{2} \left(C_2^{-1} B \right)^2 - g A^2 B \right] .$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N \delta_{p,2}$.







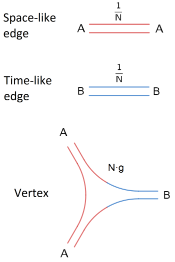
Matrix Model for the CDT Model [D. Benedetti, J. Henson, 2009]

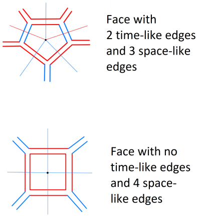
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

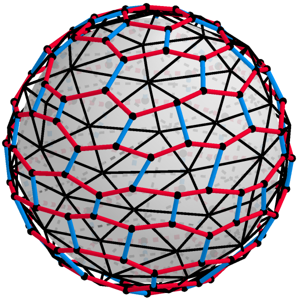
$$Z = \int dAdB\, e^{-S_{CDT}}\,, \qquad \text{where} \qquad S_{CDT} = N\,\mathrm{Tr}\left[\frac{1}{2}A^2 + \frac{1}{2}\left(C_2^{-1}B\right)^2 - gA^2B\right].$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$.







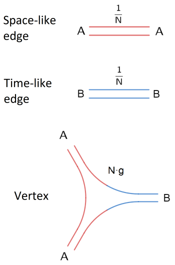
Matrix Model for the CDT Model [D. Benedetti, J. Henson, 2009]

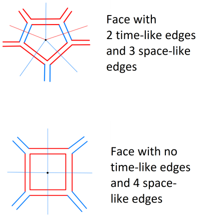
- The CDT graphs can be generated by a combinatorial analysis of the partition function:

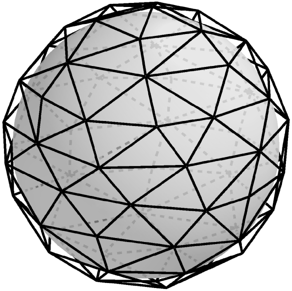
$$Z = \int dA dB \, e^{-S_{CDT}} \, , \qquad \text{where} \qquad S_{CDT} = N \, \mathrm{Tr} \left[\frac{1}{2} A^2 + \frac{1}{2} \left(C_2^{-1} B \right)^2 - g A^2 B \right] .$$

A and B: $N \times N$ Hermitian matrices.

C_2: $N \times N$ matrix. $\mathrm{Tr}\, C_2^p = N \delta_{p,2}$.







Evaluation of $\langle \chi_r(A^2) \rangle_0$

- Many integration methods are unsuitable due to the presence of the matrix C_2. We can factorize this matrix from the partition function with the use of character expansion. [D. Benedetti, J. Henson, 2009]
- Class functions can be expanded in terms of matrix characters, together with a sum over representations of $GL(N)$.

$$Z = \int dA\, e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2A^2)^2\right]}, \qquad \text{where} \qquad e^{\mathrm{Tr}M^2} = \sum_r a_r\, \chi_r(M)\,.$$

- Due to the character expansion, an integral with unknown solution appears in the calculation: The average of the character of the square of a matrix.
- *Exact* result which we can evaluate in terms of a Pfaffian of summations:

Finite N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\langle \chi_r(A^2) \rangle_0 = \frac{N^{\frac{N(N-1)}{2}}}{\prod_{m=0}^{N-1} m!} \frac{\prod_n (2h_n)!}{(2N)^{\sum_p h_p}} \operatorname*{Pf}_{i,j} \left(\sum_{\substack{k+l=2h_i \\ u+v=2h_j}} \frac{(-1)^u - (-1)^k}{2} \frac{(k+u)!!(l+v-2)!!}{k!u!l!v!} \right)$$

where $\langle Q \rangle_0 = Z^{-1} \int dA\, Q\, e^{-S[A]}$ and $h_1, \ldots, h_N$ is the set of integers defining r.

Evaluation of $\langle \chi_r(A^2) \rangle_0$

- Many integration methods are unsuitable due to the presence of the matrix C_2. We can factorize this matrix from the partition function with the use of character expansion. [D. Benedetti, J. Henson, 2009]
- Class functions can be expanded in terms of matrix characters, together with a sum over representations of $GL(N)$.

$$Z = \int dA \, e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2 A^2)^2\right]}, \qquad \text{where} \qquad e^{\mathrm{Tr} M^2} = \sum_r a_r \, \chi_r(M) \, .$$

- Due to the character expansion, an integral with unknown solution appears in the calculation: The average of the character of the square of a matrix.
- *Exact* result which we can evaluate in terms of a Pfaffian of summations:

Finite N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\langle \chi_r(A^2) \rangle_0 = \frac{N^{\frac{N(N-1)}{2}}}{\prod_{m=0}^{N-1} m!} \frac{\prod_n (2h_n)!}{(2N)^{\sum_p h_p}} \operatorname*{Pf}_{i,j} \left(\sum_{\substack{k+l=2h_i \\ u+v=2h_j}} \frac{(-1)^u - (-1)^k}{2} \frac{(k+u)!!(l+v-2)!!}{k!u!l!v!} \right)$$

where $\langle Q \rangle_0 = Z^{-1} \int dA \, Q \, e^{-S[A]}$ and $h_1, \ldots, h_N$ is the set of integers defining r.

Evaluation of $\langle \chi_r(A^2) \rangle_0$

- Many integration methods are unsuitable due to the presence of the matrix C_2. We can factorize this matrix from the partition function with the use of character expansion. [D. Benedetti, J. Henson, 2009]
- Class functions can be expanded in terms of matrix characters, together with a sum over representations of $GL(N)$.

$$Z = \int dA \; e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2 A^2)^2\right]}, \qquad \text{where} \qquad e^{\mathrm{Tr} M^2} = \sum_r a_r \, \chi_r(M)\,.$$

- Due to the character expansion, an integral with unknown solution appears in the calculation: The average of the character of the square of a matrix.
- *Exact* result which we can evaluate in terms of a Pfaffian of summations:

Finite N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\langle \chi_r(A^2) \rangle_0 = \frac{N^{\frac{N(N-1)}{2}}}{\prod_{m=0}^{N-1} m!} \frac{\prod_n (2h_n)!}{(2N)^{\sum_p h_p}} \operatorname*{Pf}_{i,j} \left(\sum_{\substack{k+l=2h_i \\ u+v=2h_j}} \frac{(-1)^u - (-1)^k}{2} \frac{(k+u)!!(l+v-2)!!}{k!u!l!v!} \right)$$

where $\langle Q \rangle_0 = Z^{-1} \int dA \, Q \, e^{-S[A]}$ and $h_1, \ldots, h_N$ is the set of integers defining r.

Evaluation of $\langle \chi_r(A^2) \rangle_0$

- Many integration methods are unsuitable due to the presence of the matrix C_2. We can factorize this matrix from the partition function with the use of character expansion. [D. Benedetti, J. Henson, 2009]
- Class functions can be expanded in terms of matrix characters, together with a sum over representations of $GL(N)$.

$$Z = \int dA \, e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2 A^2)^2\right]}, \qquad \text{where} \qquad e^{\mathrm{Tr}M^2} = \sum_r a_r \, \chi_r(M) \, .$$

- Due to the character expansion, an integral with unknown solution appears in the calculation: The average of the character of the square of a matrix.
- *Exact* result which we can evaluate in terms of a Pfaffian of summations:

Finite N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\langle \chi_r(A^2) \rangle_0 = \frac{N^{\frac{N(N-1)}{2}}}{\prod_{m=0}^{N-1} m!} \frac{\prod_n (2h_n)!}{(2N)^{\sum_p h_p}} \operatorname*{Pf}_{i,j} \left(\sum_{\substack{k+l=2h_i \\ u+v=2h_j}} \frac{(-1)^u - (-1)^k}{2} \frac{(k+u)!!(l+v-2)!!}{k!u!l!v!} \right)$$

where $\langle Q \rangle_0 = Z^{-1} \int dA \, Q \, e^{-S[A]}$ and $h_1, \ldots, h_N$ is the set of integers defining r.

Computing the eigenvalues of C_2

- Partition function and C_2:

$$Z = \int dA \, e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2 A^2)^2\right]} \qquad \text{with} \qquad \mathrm{Tr}\, C_2^p = N\delta_{2,p} \, .$$

For each $p = 1, \ldots, N$, we have an equation on the eigenvalues of C_2.

- Using the Girard–Newton formulae, we find the characteristic polynomial.
- For even N, the N eigenvalues are given by

Large N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\lambda_t^{\pm} = \pm \left[\mathrm{W}\left(-e^{\frac{4\pi}{N} i(t-1/2)-1}\right)\right]^{-\frac{1}{2}} \, , \qquad t = 1, 2, \ldots, N/2 \, .$$

$\mathrm{W}(z)$ is the Lambert function.
For N odd, the roots are the same as for $N-1$, plus 0 as an additional root.

Computing the eigenvalues of C_2

- Partition function and C_2:

$$Z = \int dA\, e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2 A^2)^2\right]} \qquad \text{with} \qquad \mathrm{Tr}\, C_2^p = N\delta_{2,p}\,.$$

For each $p = 1, ..., N$, we have an equation on the eigenvalues of C_2.

- Using the Girard–Newton formulae, we find the characteristic polynomial.

- For even N, the N eigenvalues are given by

Large N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\lambda_t^{\pm} = \pm\left[\mathrm{W}\left(-e^{\frac{4\pi}{N}i(t-1/2)-1}\right)\right]^{-\frac{1}{2}}\,, \qquad t = 1, 2, ..., N/2\,.$$

$\mathrm{W}(z)$ is the Lambert function.

For N odd, the roots are the same as for $N-1$, plus 0 as an additional root.

Computing the eigenvalues of C_2

- Partition function and C_2:

$$Z = \int dA\, e^{-N\mathrm{Tr}\left[\frac{1}{2}A^2 - \frac{g^2}{2}(C_2 A^2)^2\right]} \qquad \text{with} \qquad \mathrm{Tr}\, C_2^p = N\delta_{2,p}\,.$$

For each $p = 1, \ldots, N$, we have an equation on the eigenvalues of C_2.

- Using the Girard–Newton formulae, we find the characteristic polynomial.

- For even N, the N eigenvalues are given by

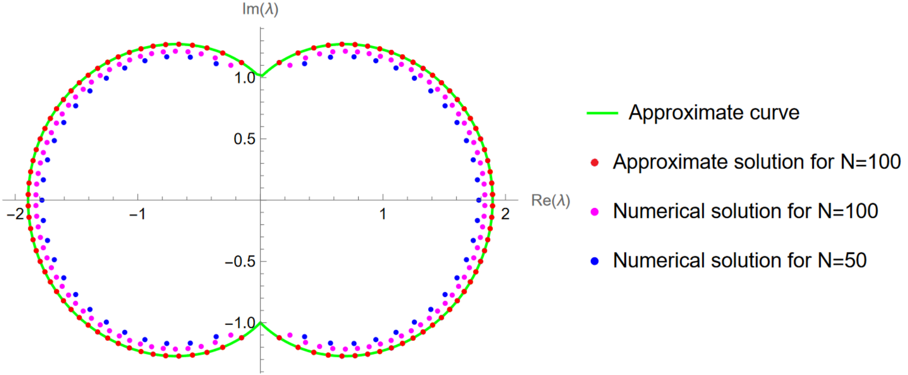
Large N result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$\lambda_t^{\pm} = \pm\left[\mathrm{W}\left(-e^{\frac{4\pi}{N}i(t-1/2)-1}\right)\right]^{-\frac{1}{2}}\,, \qquad t = 1, 2, \ldots, N/2\,.$$

$\mathrm{W}(z)$ is the Lambert function.

For N odd, the roots are the same as for $N-1$, plus 0 as an additional root.

Computing the eigenvalues of C_2



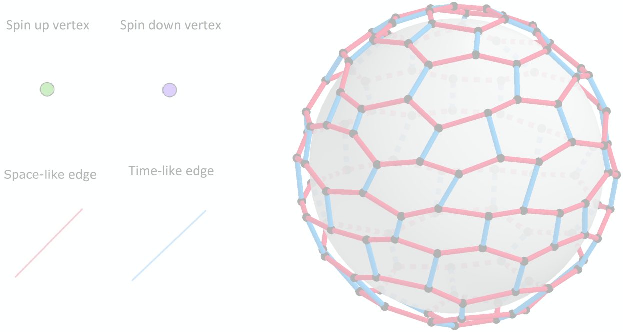
Eigenvalues λ of C_2.

CDT with Ising Model

Action for CDT coupled with the Ising Model over the vertices:

$$S_{CDTIM} = N \,\mathrm{Tr}\left[\frac{1}{2}A_+^2 + \frac{1}{2}\left({C_2}^{-1}B_+\right)^2 + \frac{1}{2}A_-^2 + \frac{1}{2}\left({C_2}^{-1}B_-\right)^2 \right.$$
$$\left. -\gamma A_+ A_- - \gamma({C_2}^{-1}B_+)({C_2}^{-1}B_-) - gA_+^2 B_+ - gA_-^2 B_-\right],$$

$A_\pm$, $B_\pm$: $N \times N$ Hermitian matrices.
C_2 : $N \times N$ matrix, $\mathrm{Tr}\, C_2^p = N\delta_{p,2}$.

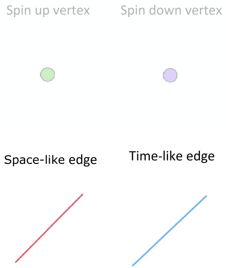


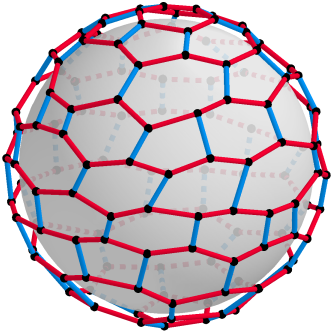
CDT with Ising Model

Action for CDT coupled with the Ising Model over the vertices:

$$S_{CDTIM} = N\,\mathrm{Tr}\Bigg[\frac{1}{2}A_+^2 + \frac{1}{2}\left({C_2}^{-1}B_+\right)^2 + \frac{1}{2}A_-^2 + \frac{1}{2}\left({C_2}^{-1}B_-\right)^2$$
$$- \gamma A_+A_- - \gamma({C_2}^{-1}B_+)({C_2}^{-1}B_-) - gA_+^2B_+ - gA_-^2B_-\Bigg] ,$$

$A_\pm$, $B_\pm$: $N \times N$ Hermitian matrices.
C_2 : $N \times N$ matrix, $\mathrm{Tr}\,C_2^p = N\delta_{p,2}$.



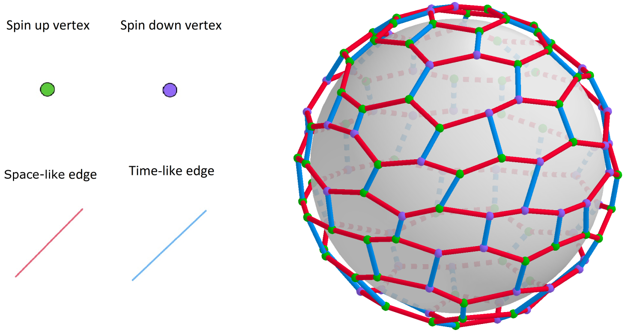


CDT with Ising Model

Action for CDT coupled with the Ising Model over the vertices:

$$S_{CDTIM} = N\,\mathrm{Tr}\Bigg[\frac{1}{2}A_+^2 + \frac{1}{2}\left(C_2{}^{-1}B_+\right)^2 + \frac{1}{2}A_-^2 + \frac{1}{2}\left(C_2{}^{-1}B_-\right)^2$$
$$- \gamma A_+A_- - \gamma(C_2{}^{-1}B_+)(C_2{}^{-1}B_-) - gA_+^2B_+ - gA_-^2B_-\Bigg],$$

$A_\pm$, $B_\pm$: $N \times N$ Hermitian matrices.
C_2 : $N \times N$ matrix, $\mathrm{Tr}\,C_2^p = N\delta_{p,2}$.



Integral over the unitary group

- Some of the integrals in the partition function can easily be solved, leading to

$$Z = \int dUdV\; e^{-N\mathrm{Tr}[\frac{1}{2}(1-\gamma)U^2+\frac{1}{2}(1+\gamma)V^2-\frac{1}{4}\frac{g^2}{1-\gamma}((U^2+V^2)C_2)^2-\frac{1}{4}\frac{g^2}{1+\gamma}((UV+VU)C_2)^2]}\,.$$

Similarly to the pure CDT, we apply the character expansion.

- The integral over the angular part between U and V can be written as

$$I = \int_{U(N)} \phi^{\alpha}_{ab}(\Omega)\phi^{\beta}_{cd}(\Omega)\overline{\phi^{\alpha}_{\bar{a}\bar{b}}(\Omega)\phi^{\beta}_{\bar{c}\bar{d}}(\Omega)}d\Omega\,, \qquad \text{with} \qquad \Omega \in U(N)\,,$$

where $\phi^{r}_{ij}(\Omega)$ are the matrix coefficients of Ω in a representation r. We use Weingarten Calculus to compute this integral.

- Expanding $\alpha \otimes \beta$ into irreducible representations r, the Clebsch-Gordan coefficients c^{rk}_{acp} appear.

Integral's result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$I = \sum_{r,m,n,p,q} c^{rm*}_{acp}\, c^{rn}_{\bar{a}\bar{c}p}\, c^{rm}_{bdq}\, c^{rn*}_{\bar{b}\bar{d}q}\, d_r^{-1}\,,$$

where d_r is the dimension of the representation r.

Integral over the unitary group

- Some of the integrals in the partition function can easily be solved, leading to

$$Z = \int dUdV \; e^{-N\mathrm{Tr}[\frac{1}{2}(1-\gamma)U^2+\frac{1}{2}(1+\gamma)V^2-\frac{1}{4}\frac{g^2}{1-\gamma}((U^2+V^2)C_2)^2-\frac{1}{4}\frac{g^2}{1+\gamma}((UV+VU)C_2)^2]} .$$

  Similarly to the pure CDT, we apply the character expansion.

- The integral over the angular part between U and V can be written as

$$I = \int_{U(N)} \phi^{\alpha}_{ab}(\Omega)\phi^{\beta}_{cd}(\Omega)\overline{\phi^{\alpha}_{\bar{a}\bar{b}}(\Omega)\phi^{\beta}_{\bar{c}\bar{d}}(\Omega)}d\Omega\,, \qquad \text{with} \qquad \Omega \in U(N)\,,$$

  where $\phi^{r}_{ij}(\Omega)$ are the matrix coefficients of Ω in a representation r. We use Weingarten Calculus to compute this integral.

- Expanding $\alpha \otimes \beta$ into irreducible representations r, the Clebsch-Gordan coefficients c^{rk}_{acp} appear.

Integral's result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$I = \sum_{r,m,n,p,q} c^{rm*}_{acp}\, c^{rn}_{\bar{a}\bar{c}p}\, c^{rm}_{bdq}\, c^{rn*}_{\bar{b}\bar{d}q}\, d_r^{-1} \,,$$

where d_r is the dimension of the representation r.

Integral over the unitary group

- Some of the integrals in the partition function can easily be solved, leading to

$$Z = \int dUdV \; e^{-N\mathrm{Tr}[\frac{1}{2}(1-\gamma)U^2+\frac{1}{2}(1+\gamma)V^2-\frac{1}{4}\frac{g^2}{1-\gamma}((U^2+V^2)C_2)^2-\frac{1}{4}\frac{g^2}{1+\gamma}((UV+VU)C_2)^2]} \,.$$

Similarly to the pure CDT, we apply the character expansion.

- The integral over the angular part between U and V can be written as

$$I = \int_{U(N)} \phi^{\alpha}_{ab}(\Omega)\phi^{\beta}_{cd}(\Omega)\overline{\phi^{\alpha}_{\bar{a}\bar{b}}(\Omega)\phi^{\beta}_{\bar{c}\bar{d}}(\Omega)}d\Omega \,, \qquad \text{with} \qquad \Omega \in U(N)\,,$$

where $\phi^{r}_{ij}(\Omega)$ are the matrix coefficients of Ω in a representation r. We use Weingarten Calculus to compute this integral.

- Expanding $\alpha \otimes \beta$ into irreducible representations r, the Clebsch-Gordan coefficients c^{rk}_{acp} appear.

Integral's result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$I = \sum_{r,m,n,p,q} c^{rm*}_{acp}\, c^{rn}_{\bar{a}\bar{c}p}\, c^{rm}_{bdq}\, c^{rn*}_{\bar{b}\bar{d}q}\, d_r^{-1} \,,$$

where d_r is the dimension of the representation r.

Integral over the unitary group

- Some of the integrals in the partition function can easily be solved, leading to

$$Z = \int dUdV\ e^{-N\mathrm{Tr}[\frac{1}{2}(1-\gamma)U^2+\frac{1}{2}(1+\gamma)V^2-\frac{1}{4}\frac{g^2}{1-\gamma}((U^2+V^2)C_2)^2-\frac{1}{4}\frac{g^2}{1+\gamma}((UV+VU)C_2)^2]}\,.$$

  Similarly to the pure CDT, we apply the character expansion.

- The integral over the angular part between U and V can be written as

$$I = \int_{U(N)} \phi^{\alpha}_{ab}(\Omega)\phi^{\beta}_{cd}(\Omega)\overline{\phi^{\alpha}_{\bar{a}\bar{b}}(\Omega)\phi^{\beta}_{\bar{c}\bar{d}}(\Omega)}d\Omega\,, \qquad \text{with} \qquad \Omega \in U(N)\,,$$

  where $\phi^{r}_{ij}(\Omega)$ are the matrix coefficients of Ω in a representation r. We use Weingarten Calculus to compute this integral.

- Expanding $\alpha \otimes \beta$ into irreducible representations r, the Clebsch-Gordan coefficients c^{rk}_{acp} appear.

Integral's result [J. L. A. Abranches, A. D. Pereira, R. Toriumi, 2024]

$$I = \sum_{r,m,n,p,q} c^{rm*}_{acp}\, c^{rn}_{\bar{a}\bar{c}p} c^{rm}_{bdq} c^{rn*}_{\bar{b}\bar{d}q} d_r^{-1}\ ,$$

where d_r is the dimension of the representation r.

Thank you!