Duality and double scaling limit of random tensor models

Thomas MULLER

Based on : Kepler and Muller, Lett Math Phys 113, 83 (2023)
Keppler, Krajewski, Muller, Tanasa J. Phys. A : Math. Theor. 56 495206
T Krajewski, T Muller, A Tanasa 2023 J. Phys. A : Math. Theor. 56 235401

LABRI, Univ. Bordeaux

Random tensors 2024

Random matrix models

Matrix M (of size N) whose components are random variables :

$$Z = \int [dM] e^{-Tr(M^2)+\lambda Tr(M^n)}, \quad \langle P(M) \rangle = \frac{1}{Z} \int [dM] P(M) e^{-Tr(M^2)+\lambda Tr(M^n)}$$

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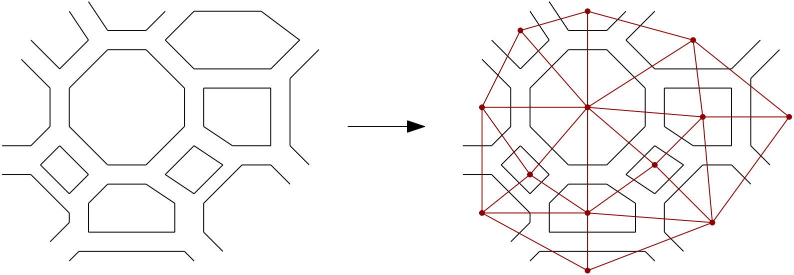
Diagrammatic expansion : Feynman ribbon graphs, dual to triangulations of surfaces



Matrix large N expansion and double scaling limit

Large N expansion : rearrange terms in powers of N (controlled by genus g of the surface) :

$$Z = \sum_g N^{2-2g} A_g(\lambda)$$

A_g : contribution of surfaces with genus g
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Double scaling limit : For higher genus, there is a critical value λ_c for which

$$A_g(\lambda) \approx (\lambda - \lambda_c)^{-\alpha(g)}$$

Taking $\lambda \to \lambda_c$ enhance non planar surfaces !

- take $N \to \infty$ and $\lambda \to \lambda_c$ such that both effects cancel out.
- obtain contribution from any genus.
- key mechanism : related to continuum limit of the model

Random tensor models

(See Klebanov, Popov, Tarnopolsky : TASI lectures on tensor models, Tanasa : Combinatorial physics, Gurau : Random Tensors)

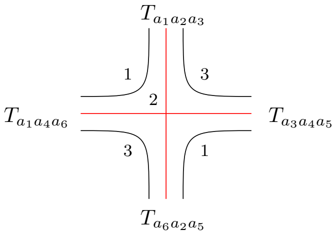
Arise as generalization of matrix models in higher dimensions :

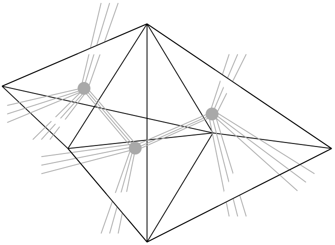
$$Z = \int [dT] e^{-S[T]}, \quad \langle P(T) \rangle = \frac{1}{Z} \int [dT] e^{-S[T]} P(T)$$

$$S[T] = T^{a_1 \dots a_D} C_{a_1 \dots a_D b_1 \dots b_D} T^{b_1 \dots b_D} - \sum_{\mathcal{B}} \lambda_{\mathcal{B}} I_{\mathcal{B}}(T).$$

Ribbon graphs → $d-$stranded graphs (or $d+1$-edge colored graphs)

2 dimensional triangulations → d dimensional triangulations





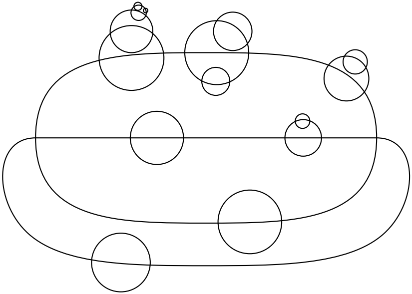
Different behavior than matrix models !

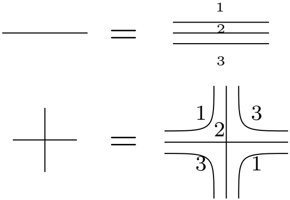
Large N limit of tensor models

Large N expansion now controlled by gurau's degree :

$$Z = \sum_{\omega} N^{3-\omega} A_{\omega}$$

Dominated by melon diagrams :

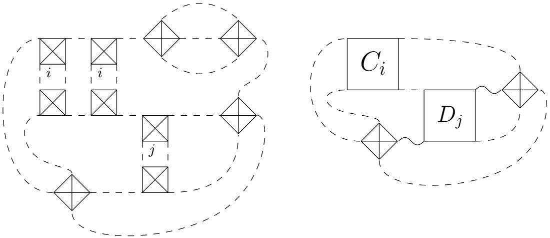




Simpler than planar limit of matrix models but richer than large N limit of vector models.

Double scaling limit for tensors models

Mostly studied in the quartic case.
Not all the graphs contribute in the double scaling limit, only the one called dominant.
Can be identified using the scheme decomposition (Chapuy & al. 2007, Gurau & Schaeffer, 2013) :



Schemes of dominant graphs are in bijection with rooted binary trees.
(V Bonzom, V Nador, A Tanasa 2021 or A Tanasa, E Fusy 2014 etc...).

Motivation

Objective of my work : study combinatorial aspects of random tensor models.

In this talk :

- Double scaling limit (DSL) of the prismatic tensor models.
    - model with non standard large N limit (new double scaling limit ?).
    - Generalize DSL implementation to sextic interactions

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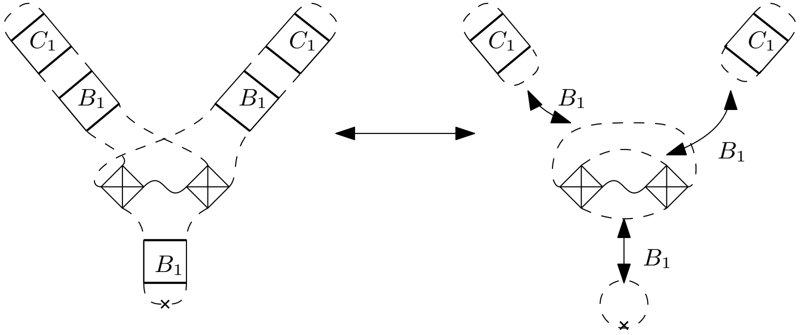
  - model with non standard large N limit (new double scaling limit ?).
  - Generalize DSL implementation to sextic interactions
- Duality of orthogonal and symplectic tensor models.
  - Generalize the work of Gurau and Keppler (arXiv :2207.01993) to arbitrary interactions and symmetries
  - Try new tools to study tensors models with irreducible symmetries. (brauer algebra)

Double scaling limit of the prismatic tensor model

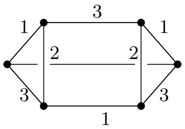


The prismatic tensor model

Model with symmetry $O(N)^3$ and sextic interaction :

$$S = \frac{1}{2}\sum_{abc} T_{abc} T_{abc} - \frac{\lambda N^{-3}}{6} I_p(T)$$

$$I_p(T) = T_{a_1 b_1 c_1} T_{a_1 b_2 c_2} T_{a_2 b_1 c_2} T_{a_3 b_3 c_1} T_{a_3 b_2 c_3} T_{a_2 b_3 c_3}$$



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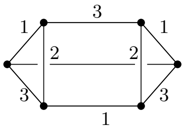
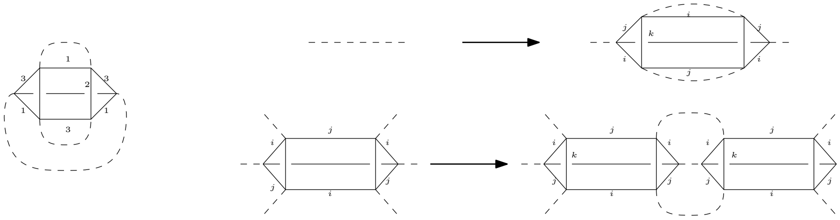


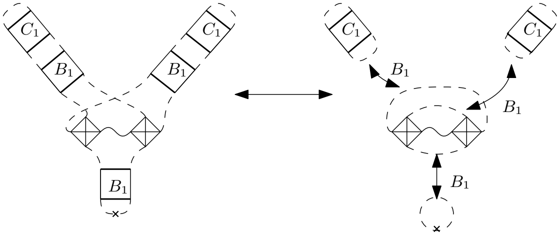
New move in the large N limit :



Double scaling - Results

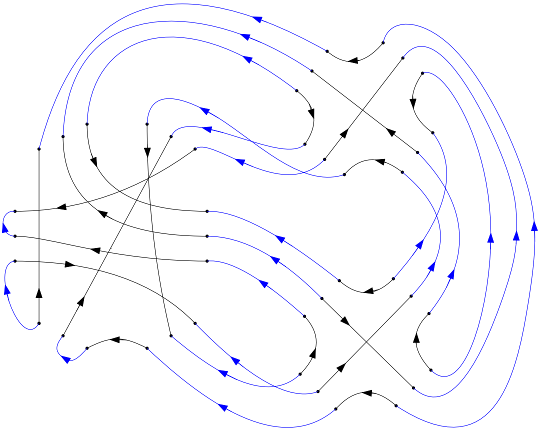
T Krajewski, T Muller, A Tanasa 2023 J. Phys. A : Math. Theor. 56 235401

Theorem : **The schemes of the (rooted) dominant graphs are in bijection with rooted binary trees with $2\omega - 1$ edges, $\omega - 1$ internal nodes given by degree 0 components and ω leaves given by degree 1 components.**



Proof is more technical but similar results than in the quartic case.

Duality of orthogonal and symplectic tensor models



Orthogonal and symplectic tensor models

Consider the two groups :

- $O(N)$: isometry group of the standard symmetric form $\delta = \mathbb{1}_N$.
  If $M\delta M^T = \delta$, $M \in O(N)$
- $Sp(N)$: isometry group of the standard symplectic form ω :

$$\omega = \left(\begin{array}{c|c} 0 & \mathbb{1}_{N/2} \\ \hline -\mathbb{1}_{N/2} & 0 \end{array}\right)$$

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There is a well known relation linking these two groups, stated as $O(-N) \approx Sp(N)$.
Occurs in representation theory, gauge theories, vector models etc...

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Was proven by Gurau and Keppler that this relation also occurs for tensor models with quartic interactions.

We generalised this proof to arbitrary interactions and irreps of $O(N)$ and $Sp(N)$.

Duality in tensor models

Duality for Colored tensor models :

H Keppler, T Muller Lett.Math.Phys. 113 (2023) 4, 83

Theorem : Tensor models with no symmetry properties under permutation of their indices and invariance given by the tensor product of different copies of $O(N)$ or $Sp(N)$ are dual to tensor models obtained by changing the symmetry of some given colors.

ex : $O(N_1) \otimes O(N_2) \otimes O(N_3)$ and $O(N_1) \otimes Sp(N_2) \otimes Sp(N_3)$

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Similar results for tensors with symmetries of their indices (irreps of $O(N)$ and $Sp(N)$)

H Keppler, T Krajewski, A Tanasa, T Muller J.Phys.A 56 (2023) 49, 495206

General result ! Don't depend on choice of trace invariants/ rank of the tensors

Conclusion

Two aspects of random tensor models were studied here :

- The double scaling limit
- Duality between the $O(N)$ and $Sp(N)$ symmetry

The results found in the double scaling limit of the prismatic models are similar to the ones found for tensor models with quartic interactions.

We have also proven the duality for tensors with no symmetries properties of their indices and for ones transforming under irreducible representations of $O(N)$ and $Sp(N)$.

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Perspectives :

- Study double scaling limit with all sextic interactions (non tree-like dominant graphs using the wheel interaction ?)
- Study sextic field theories with $O(N)^3$ symmetries and non trivial power of the propagator (current work of Gaëtan Eliott Bardy)
- Investigate some "$N \to 0$" limit of tensor models using the proven duality.