# Signed eigenvalue distributions of complex random tensors and geometric measure of entanglement of multipartite states

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Mainly based on

S. Majumder, NS, PTEP 2024 (2024) 9, 093A01, arXiv:2408.01030 [hep-th]
N. Delporte, NS, arXiv:2405.07731 [hep-th]
NS, PTEP 2024 (2024) 5, 053A04, arXiv:2404.03385 [hep-th]
M.R. Kloos, NS, Lett.Math.Phys. 114 (2024) 3, 80, arXiv:2403.12427 [hep-th]

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Random tensors and related topics
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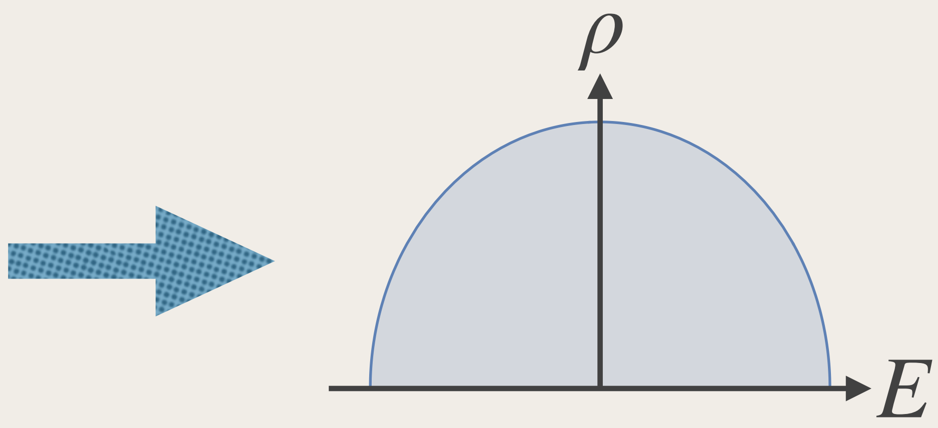
# §Introduction

**Eigenvalue distributions are important in random matrix models**

- Approximate Hamiltonian of atoms (Wigner 1958)

$H$ : Random matrix



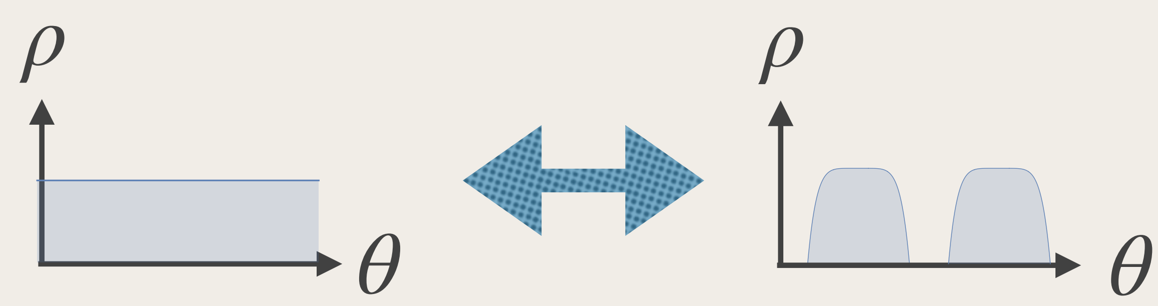

Semi-circle law

- Method of computing matrix models

Brezin-Itzykson-Parisi-Zuber 1978

- Topological transition — Dynamics of QCD

Gross-Witten, Wadia, 1980

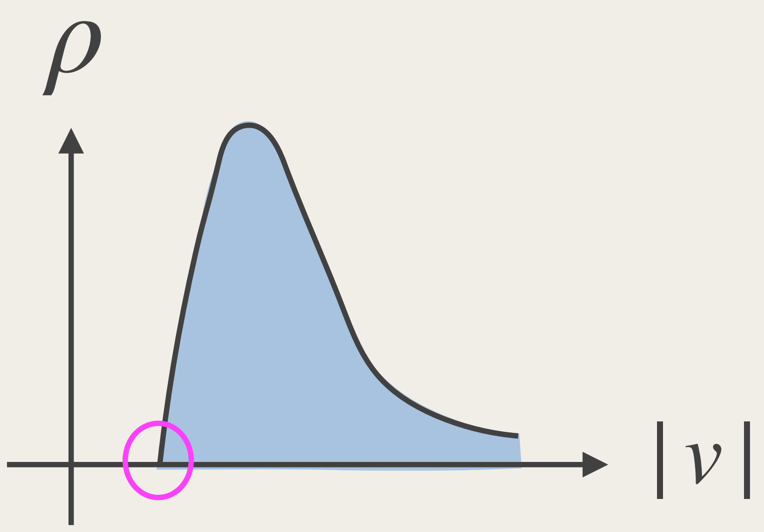


## How about tensor eigenvalue distributions ?

Most tensor problems are NP-hard for a tensor. Hillar-Lim 2009

On the other hand, a distribution of tensor eigenvalues for an ensemble of tensors can exactly/approximately computed, as we will do by using quantum field theories

In $N \to \infty$ a sharp edge of the distribution appears, which is important, since it determines the "best" value in applications.




- Ground state energy of spin glass
- Largest eigenvalue
- Best rank-one decomposition of tensor
- Geometric measure of entanglement of multipartite states

# §Geometric measure of entanglement of multipartite states

- **Bipartite state** $|\psi\rangle = M_{ab}|a\rangle_A|b\rangle_B$ $\quad |a\rangle_A \in H_A \quad |b\rangle_B \in H_B$

Entanglement entropy

$$S = -\mathrm{Tr}_A(\Omega_A \log \Omega_A) = -\mathrm{Tr}_B(\Omega_B \log \Omega_B)$$

$$\Omega_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$$
$$\Omega_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$$

- **Tripartite state**

$$|\psi\rangle = C_{abc}|a\rangle_A|b\rangle_B|c\rangle_C \qquad |a\rangle_A \in H_A \quad |b\rangle_B \in H_B \quad |c\rangle_C \in H_C$$

Generally,

$$-\mathrm{Tr}_A(\Omega_A \log \Omega_A) \neq -\mathrm{Tr}_B(\Omega_B \log \Omega_B)$$

$$\Omega_A = \mathrm{Tr}_{BC}(|\psi\rangle\langle\psi|)$$
$$\Omega_B = \mathrm{Tr}_{AC}(|\psi\rangle\langle\psi|)$$

How can we measure entanglement of multipartite states ?

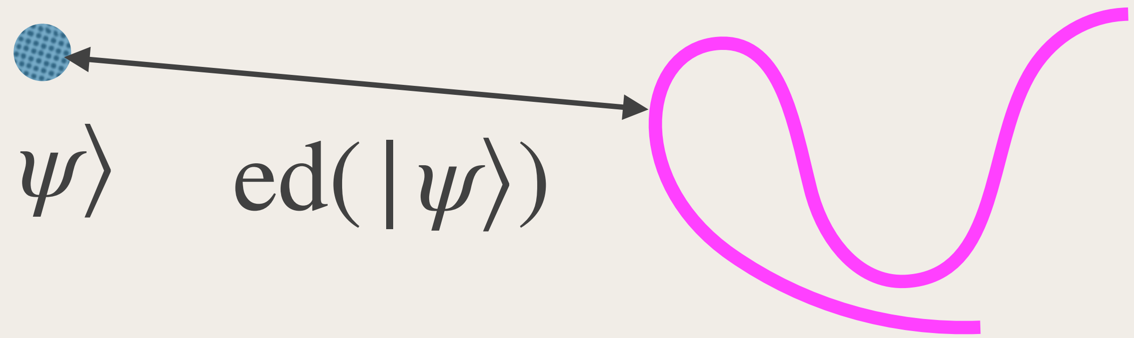
# §Geometric measure of entanglement of multipartite states

Shimony 1995, Barnum-Linden 2001, Wei-Goldbart 2003

The amount of entanglement may be measured by the minimum distance from product states.

Ex. Tripartite states

$$\mathrm{ed}(|\psi\rangle) = \min_{\psi_{A,B,C}} \left| |\psi\rangle - |\psi_A\rangle_A \otimes |\psi_B\rangle_B \otimes |\psi_C\rangle_C \right|$$



## Representation in tensor

$$|\psi\rangle = C_{abc}|a\rangle_A \otimes |b\rangle_B \otimes |c\rangle_C$$

$$|C|^2 = |v^{(A,B,C)}|^2 = 1$$

$$|\psi_A\rangle_A = v_a^{(A)}|a\rangle_A$$

$$|\psi_B\rangle_B = v_b^{(B)}|b\rangle_B$$

$$|\psi_C\rangle_C = v_c^{(C)}|c\rangle_C$$

$$\mathrm{ed}(|\psi\rangle)^2 = \min_{\psi_{A,B,C}} \Big|\, |\psi\rangle - |\psi_A\rangle_A \otimes |\psi_B\rangle_B \otimes |\psi_C\rangle_C \Big|^2$$

$$= 2 - 2 \max_{v_{a,b,c}^{(A,B,C)}} \mathrm{Re}[C^*_{abc} v_a^{(A)} v_b^{(B)} v_c^{(C)}]$$

← Injective norm

$$\frac{\partial\, \mathrm{ed}(|\psi\rangle)}{\partial v_{a,b,c}^{(A,B,C)}} = 0 \quad \Rightarrow \quad \begin{cases} C^*_{abc} v_b^{(B)} v_c^{(C)} = v_a^{(A)*} \\ C^*_{abc} v_a^{(A)} v_c^{(C)} = v_b^{(B)*} \\ C^*_{abc} v_a^{(A)} v_b^{(B)} = v_c^{(C)*} \end{cases}$$

A system of eigenvector equations

ed($|\psi\rangle$) is determined by the eigenvector of smallest $|v| = |v_i|$.
→The edge of the eigenvector distribution determines the geometric measure of entanglement of random multipartite states.

# § Complex eigenvector problems

S. Majumder, NS, PTEP 2024 (2024) 9, 093A01, arXiv:2408.01030 [hep-th]
NS, PTEP 2024 (2024) 5, 053A04, arXiv:2404.03385 [hep-th]

We compute the distributions of eigenvectors of complex order-three random tensors with symmetric or independent indices.

- **Symmetric**

$$C_{abc} = C_{\sigma_a \sigma_b \sigma_c},\ v_a \in \mathbb{C} \ \ (\sigma : \text{arbitrary perms. of } a, b, c)$$

$$C^*_{abc} v_b v_c = v^*_a \qquad : \text{Eigenvector equation}$$

Corresponds to $|\psi\rangle = C_{abc} |a\rangle \otimes |b\rangle \otimes |c\rangle$

- **Independent indices**

$$C_{abc}, v_a^{(A)}, v_b^{(b)}, v_c^{(C)} \in \mathbb{C}$$

$$\begin{cases} C^*_{abc} v_b^{(B)} v_c^{(C)} = v_a^{(A)*} \\ C^*_{abc} v_a^{(A)} v_c^{(C)} = v_b^{(B)*} \\ C^*_{abc} v_a^{(A)} v_b^{(B)} = v_c^{(C)*} \end{cases} \quad : \text{A system of eigenvector equations}$$

Corresponds to $|\psi\rangle = C_{abc} |a\rangle_A \otimes |b\rangle_B \otimes |c\rangle_C$

# § Field theoretical method

cf. A. Crisanti, L. Leuzzi, and T. Rizzo, Eur. Phys. J. B 36, 129-136 (2003)

## General form of the problem

Number of d.o.f. of $v$

$$f_i(v, C) = 0 \quad : \text{linear in } C \qquad i = 1,2,\cdots,\#v$$

## Eigenvector distribution for a Gaussian ensemble of the tensor $C$

$$\rho(v) = \int dC \, e^{-\alpha C^*_{abc} C_{abc}} \, |\det M(v, C)| \prod_{i=1}^{\#v} \delta(f_i(v, C)) \qquad M(v, C)_{ij} = \frac{\partial f_j}{\partial v_i}$$

Bosons + Fermions

Jacobian

## Signed eigenvector distribution easier to compute

$$|\det M| \longrightarrow \det M = \int d\bar{\psi} d\psi \, e^{\bar{\psi} M \psi} \qquad \bar{\psi}, \psi : \text{Fermions only}$$

## After integrating over $C$

$$\rho_{\text{signed}}(v) = \mathcal{N}' \int d\bar{\psi} d\psi \, e^{S_{ff}} \qquad S_{ff} : \text{Four-fermi action}$$

# Four-fermi actions

- **Symmetric indices case**

$$S_{ff} = \bar{\psi} \cdot \psi + \bar{\varphi} \cdot \varphi + \frac{2|v|^2}{3\alpha} \left( \bar{\psi} \cdot \varphi \, \bar{\varphi} \cdot \psi - \bar{\psi} \cdot \psi \, \bar{\varphi} \cdot \varphi \right) + \text{parallel to } v, v^*$$

- **Independent indices case**

$$S_{ff} = \sum_{i=1}^{3} \left( \bar{\psi}_i \cdot \psi_i + \bar{\varphi}_i \cdot \varphi_i \right) + \frac{|v|^2}{\alpha} \sum_{i<j}^{3} \left( \bar{\psi}_i \varphi_j + \bar{\psi}_j \varphi_i \right) \cdot \left( \bar{\varphi}_i \psi_j + \bar{\varphi}_j \psi_i \right)$$

$$+ \text{parallel to } v, v^*$$

The partition function of these four-fermi theories can exactly be computed by using the following type of manipulations:

$$e^{g \, \bar{\psi} \cdot \psi \, \bar{\varphi} \cdot \varphi} = e^{g \frac{\partial}{\partial k_1} \frac{\partial}{\partial k_2}} e^{k_1 \, \bar{\psi} \cdot \psi + k_2 \, \bar{\varphi} \cdot \varphi} \Big|_{k_1 = k_2 = 0}$$

Exact closed-form expressions are given in terms of generating functions.

- **Symmetric indices case**

$$\rho_{\text{signed}}(|v|^2) = -3^N \alpha^N |v|^{-2N-2} e^{-\frac{\alpha}{|v|^2}} (1+g\,l)^{-2} \exp\left(\frac{l}{1+g\,l}\right)\Bigg|_{l^{N-1}}$$

$$g = 2|v|^2/(3\alpha)$$

Taking the $l^{N-1}$-th order

- **Independent indices case**

$$\rho_{\text{signed}}(|v|^2) = -\alpha |v|^{-4} e^{-\frac{\alpha}{|v|^2}} (1-t_2+2t_3)^{-2} \exp\left(\frac{t_1-2t_2+3t_3}{g(1-t_2+2t_3)}\right)\Bigg|_{\prod_{i=1}^{3} l_i^{N_i-1}}$$

$$g = |v|^2/\alpha$$

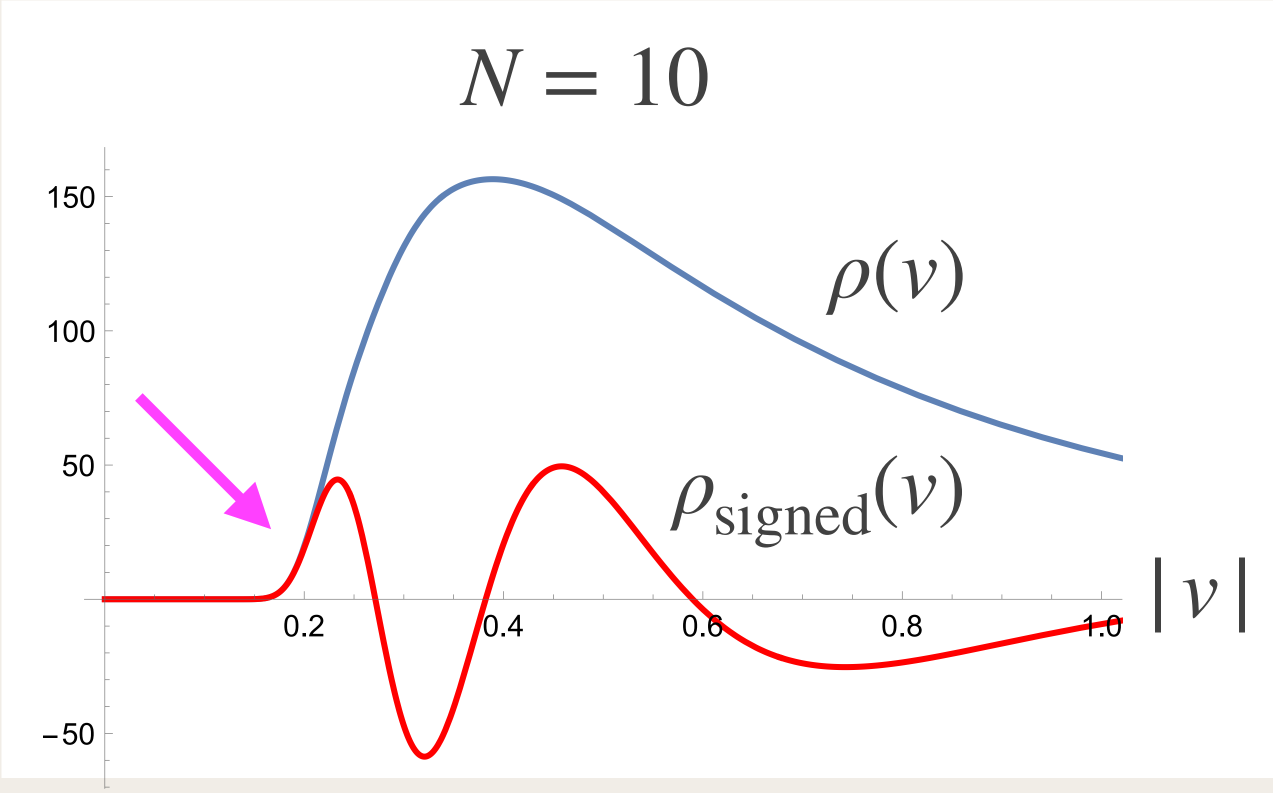
$N_i$ : dimension of $i$-th index

$$t_1 = l_1 + l_2 + l_3$$
$$t_2 = l_1 l_2 + l_2 l_3 + l_3 l_2$$
$$t_3 = l_1 l_2 l_3$$

# Location of edge can be derived from the signed distribution

M.R. Kloos, NS, Lett.Math.Phys. 114 (2024) 3, 80, arXiv:2403.12427 [hep-th]

Ex. Real eigenvector distribution of real symmetric random tensor




Large N asymptotic forms of the genuine and signed distributions are expressed by the same function $h$, and hence have a common edge.

$$\rho(v) \sim e^{N\,\mathrm{Re}[h(v)]}$$

$$\rho_{\text{sigend}}(v) \sim \mathrm{Re}[e^{N\,h(v)}]$$

$$h(v_{\text{edge}}) = 0$$

In the current cases, the asymptotic forms in the large N limit can be extracted from the exact closed-form expressions.

$$\rho_{\text{signed}}(|v|^2) \sim \text{Re}\left[e^{N h(|v|)}\right]$$

The locations of the edges are computed by solving $h(|v|_{\text{edge}}) = 0$

**Symmetric indices case** $C_{abc} \sim N\left(0, 1/\sqrt{2\alpha}\right) \times (\text{sym. fac.})$

$$|v|_{\text{edge}} = 0.603501\sqrt{\frac{\alpha}{N}}$$

**Independent indices case with** $N_i = N$

$$|v|_{\text{edge}} = 0.348431\sqrt{\frac{\alpha}{N}}$$

# § Agreement with a pervious numerical study

K. Fitter, C. Lancien, I. Nechita, "Estimating the entanglement of random multipartite quantum states," [arXiv:2209.11754 [quant-ph]]

**Symmetric indices case** $\left(|C|_{\text{inj}} = \max_{|w|=1} C_{abc} w_a w_b w_c\right)$

$$|C|_{\text{inj}} = 1/|v|_{\text{edge}} = 2.34335 \quad (\alpha = N/2)$$

FLN result = 2.356248 Error~0.5%

**Independent indices case** $\left(|C|_{\text{inj}} = \max_{|w^i|=1} C_{abc} w_a^1 w_b^2 w_c^3\right)$

$$|C|_{\text{inj}} = 1/|v|_{\text{edge}} = 4.0588 \quad (\alpha = N/2)$$

FLN result = 4.143529 Error~2%

The numbers can be regarded as being coincident, since the errors are smaller than the 4% for the established case (real case).

# §Summary

As in matrix models, tensor eigenvalue/vector distributions may become important in various applications.

The quantum field theoretical method is a powerful practical method of computing them.

In particular signed distributions are the easiest but useful, and can be computed by four-fermi theory.

We have computed the signed eigenvalue/vector distributions of complex random tensors, and have derived the asymptote of the geometric measure of quantum entanglement analytically for the first time. (cf. Dartois, McKenna, arXiv:2404.03627)

**Future prospects**

The study of tensor eigenvalue/vector distributions is rather new, and there will be more developments and applications.

There will be extensions, such as tensor rank decomposition, etc.

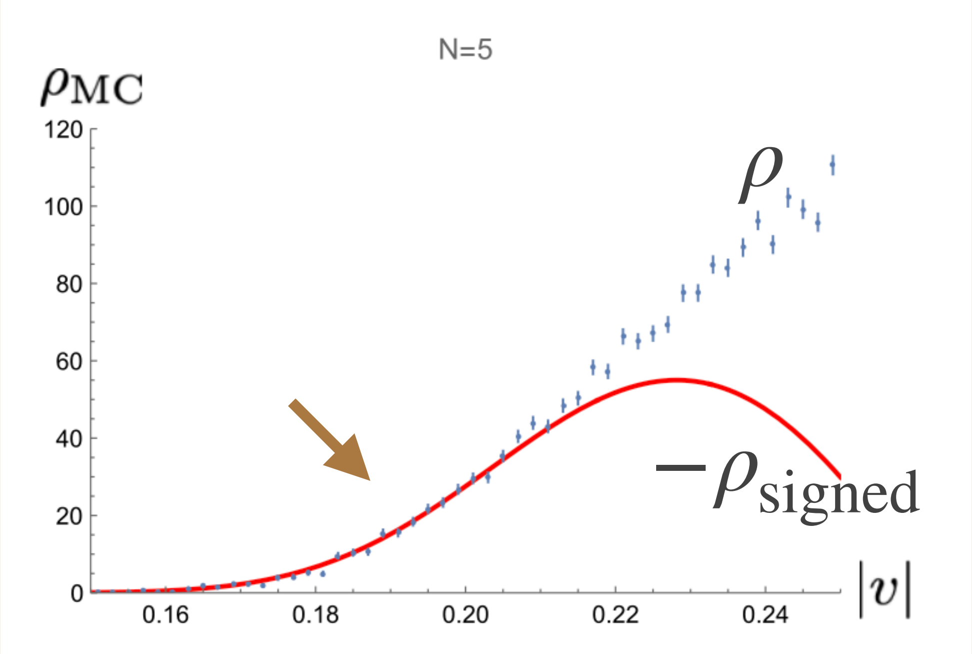
# Thank you !

Merci !

# § Checked with Monte Carlo simulations

The signed distribution agrees with the genuine distribution !

**Symmetric indices case** $N = 5$




**Independent indices case** $(N_1, N_2, N_3) = (3,2,2)$