Tensor freeness and central limit theorem

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joint work with Ion Nechita [1]

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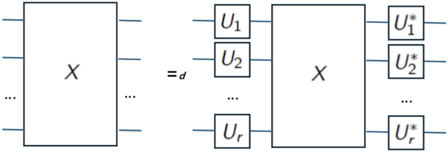
Random tensors, IHP, October 2nd, 2024

Overview

We consider random matrices $X = X_N \in M_N(\mathbb{C})^{\otimes r} \cong M_{N^r}(\mathbb{C})$ satisfying the local-unitary invariance (local-UI):

$$X =_d UXU^*, \quad \forall\, U = \bigotimes_{s=1}^r U_s \in \mathcal{U}_N^{\otimes r}.$$



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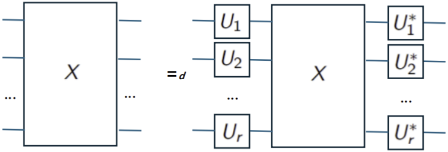

Main question

What can be expected about the convergences of local-UI random matrices in the limit $N \to \infty$? For example,

$$X_N, Y_N\text{: indep, local-UI, each having a limit} \implies \exists \text{ limit of } X_N + Y_N?$$

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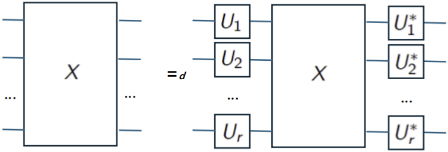

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$\longrightarrow$ Answered via "$\otimes$-free probability"!

Motivation: Unitary invariance and asymptotic freeness

Theorem (Voiculescu 1991 & 1998; Collins 2003)

$X_N^{(1)}, \ldots, X_N^{(L)}$: *independent* $N \times N$ *random matrices such that*

- *each* $X_N^{(i)}$ *is unitary invariant,*
- *each* $X_N^{(i)}$ *converges in (non-commutative) distribution,*
- *factorization property: for* $i \in [L]$ *and polynomials* P_j,
  $\mathbb{E}[\mathrm{tr}(P_1(X_N^{(i)})) \cdots \mathrm{tr}(P_r(X_N^{(i)}))] = \prod_{j=1}^{r} \mathbb{E}[\mathrm{tr}(P_j(X_N^{(i)}))] + o(1)$.

Then $X_N^{(1)}, \ldots X_N^{(L)}$ *are asymptotically freely independent as* $N \to \infty$.

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- The asymptotic freeness remains valid for
  - (Dykema 1993) Wigner matrices,
  - (Collins and Sniady 2006) Orthogonal invariant matrices.
- (Male 2020; Cebron, Dahlqvist, and Male 2024) Permutation invariant matrices and traffic independence.

Result: Local unitary invariance and $\otimes$-freeness

Theorem (Nechita and P., in progress)

$X_N^{(1)}, \dots, X_N^{(L)}$: *independent families of* $N^r \times N^r$ *random matrices s.t.*

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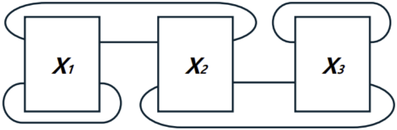
- $\otimes$-distribution = data of $\left(\mathbb{E} \circ \mathrm{tr}_{\underline{\alpha}} := \frac{1}{N^{\#\alpha_1 + \cdots + \#\alpha_r}} \mathbb{E} \circ \mathrm{Tr}_{\underline{\alpha}}\right)_{\underline{\alpha} \in \bigcup_{p=1}^{\infty} (S_p)^r}$ for a family of random matrices $\mathcal{W}_N \subset M_{N^r}(L^{\infty-}(\mathbb{P}))$.
- $\mathrm{Tr}_{(12)(3),(1)(23)}(X_1, X_2, X_3) =$



$\otimes$-Free cumulants and $\otimes$-free independence

Definition (Nechita and P., in progress)

An (r-partite) algebraic $\otimes$-probability space is a triple $(\mathcal{A}, \varphi, (\varphi_{\underline{\alpha}}))$ where

- *$\mathcal{A}$ is a unital algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional,*
- *for each $p \geq 1$ and $\underline{\alpha} \in (S_p)^r$, $\varphi_{\underline{\alpha}} : \mathcal{A}^p \to \mathbb{C}$ is a multilinear functional satisfying "reasonable conditions".*

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1. *Define $\otimes$-free cumulants $\kappa_{\underline{\alpha}} : \mathcal{A}^p \to \mathbb{C}$ for $\underline{\alpha} \in (S_p)^r$ by*

$$\varphi_{\underline{\alpha}}(x_1, \dots, x_p) = \sum_{\underline{\beta}:\, |\beta_s| + |\beta_s^{-1}\alpha_s| = |\alpha_s|\, \forall\, s} \kappa_{\underline{\beta}}(x_1, \dots, x_p).$$

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2. *Unital subalgebras $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(L)}$ of $\mathcal{A}$ are called $\otimes$-freely independent if every mixed $\otimes$-free cumulant vanishes, i.e., $\kappa_{\underline{\alpha}}(x_1, \ldots, x_p) = 0$ whenever for some s, α_s connects elements from different algebras.*

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2. $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(L)}$ *are* *$\otimes$-freely independent* *if* $\kappa_{\underline{\alpha}}(x_1, \dots, x_p) = 0$ *whenever for some* s, α_s *connects elements from different algebras.*

REMARK:

- $$\kappa_{\underline{\alpha}}(x_1, \dots, x_p) := \sum_{\underline{\beta}\colon |\beta_s| + |\beta_s^{-1}\alpha_s| = |\alpha_s| \, \forall s} \varphi_{\underline{\beta}}(x_1, \dots, x_p) \left(\prod_{s=1}^{r} \text{Möb}(\beta_s^{-1}\alpha_s) \right).$$
- If $r = 1$, $\kappa_{\underline{\alpha}} = \kappa_{\alpha}$ and free $=$ $\otimes$-free.
- $\otimes$-Freeness $\longrightarrow$ the joint $\otimes$-distribution depends only on the marginal $\otimes$-distributions.

Result 1: Local unitary invariance and $\otimes$-freeness

Theorem (Nechita and P., in progress)

$X_N^{(1)}, \ldots, X_N^{(L)}$: *independent families of $N^r \times N^r$ random matrices s.t.*

- *each $X_N^{(i)}$ is local-UI,*
- *each $X_N^{(i)}$ convergies in $\otimes$-distribution, i.e., the limit*

$$\lim_{N\to\infty} \mathbb{E}\left[\mathrm{tr}_{\underline{\alpha}}(X_N^{(i)}, \ldots, X_N^{(i)})\right]$$

exists for all $i = 1, \ldots, L$ and $\underline{\alpha} \in (S_p)^r$,

- *+ "factorization property".*

Then $X_N^{(1)}, \ldots X_N^{(L)}$ are asymptotically $\otimes$-free as $N \to \infty$.

Result 2: CLT for $\otimes$-free elements

Theorem (Voiculescu 1986)

Let $\{x_i\}_{i=1}^{\infty} \subset (\mathcal{A}, \varphi)$ be a family of centered, identically distributed, and free elements. Then $\frac{1}{\sqrt{N}}(x_1 + \cdots + x_N)$ converges in distribution to a semicircular element of variance $\varphi(x_1^2)$. Equivalently,

$$\kappa_p\left(\frac{x_1 + \cdots + x_N}{\sqrt{N}}\right) \to \begin{cases} \kappa_2(x_1) = \varphi(x_1^2) & \text{if } p = 2, \\ 0 & \text{otherwise.} \end{cases}$$

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Theorem (Nechita and P., in progress)

Let $\{x_i\}_{i=1}^{\infty} \subset (\mathcal{A}, \varphi, (\varphi_{\underline{\alpha}}))$ *be a family of identically* $\otimes$*-distributed and* $\otimes$*-free elements. Then as* $N \to \infty$,

$$\kappa_p\left(\frac{x_1+\cdots+x_N - N\varphi(x_1)}{\sqrt{N}}\right) \to \begin{cases} \sum_{\underline{\alpha}} \kappa_{\underline{\alpha}}(x_1) & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd,} \end{cases}$$

where the sum in the first case is taken over $\alpha_1, \ldots, \alpha_r \in NC_{1,2}(p)$ *such that* $\bigvee_{NC} \alpha_i = 1_p$ *and* $\bigvee_{\mathcal{P}} \alpha_i \in \mathcal{P}_2(p)$.

Application: CLT for tensor products of free variables

Suppose that $\{a_i\}_{i=1}^{\infty} \subset (\mathcal{A}, \varphi)$ is a family of self-adjoint, identically distributed free elements with mean $\varphi(a_i) = \lambda$ and variance $\mathrm{var}(a_i) = \sigma^2$. Consider the identically $\otimes$-distributed and $\otimes$-free elements

$$x_i := a_i \otimes a_i \in (\mathcal{A}^{\otimes 2}, \varphi^{\otimes 2}, (\varphi_{\alpha_1} \otimes \varphi_{\alpha_2})).$$

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Corollary (Lancien, Santos, and Youssef 2024+; Skoufranis 2024+)

$$\lim_{N\to\infty} \kappa_{2p}\left(\frac{x_1+\cdots+x_N-N\lambda^2}{\sqrt{N}}\right) = \begin{cases} \sigma^4 + 2\sigma^2\lambda^2 & \text{if } p = 1, \\ 2M_p\sigma^{2p}\lambda^{2p} & \text{if } p > 1, \end{cases}$$

where M_p is the cardinality of a set depending only on p.

Outlook and future works

In this work, we provided

- Convergence of local-UI random matrices in the framework of $\otimes$-free probability.
- CLT for $\otimes$-free elements and its applications.
- Further results: asymptotic freeness and $\otimes$-freeness of non-indep. random matrices (e.g. partial transposes of UI matrices).

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Future works:

- Comparision with other notions, e.g.
  - Tensor cumulants by Kunisky, Moore, and Wein (2024+),
  - Freeness of tensors by Bonnin and Bordenave (2024+).
- Any possible application to Quantum Information Theory.

Thank you for your attention!