Truncated HOSVD: A Random Matrix Analysis

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Random Tensors and Related Topics

Spiked Random Tensor Model

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$$\mathfrak{T} = \qquad \mathcal{P} \qquad + \quad \frac{1}{\sqrt{N}}\mathfrak{N} \quad \in \mathbb{R}^{n_1 \times n_2 \times n_3}, \qquad \begin{array}{l} N = n_1 + n_2 + n_3 \\ \mathfrak{N}_{i,j,k} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1) \end{array}$$

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$\rightarrow$ Semi-orthogonal matrices $n_\ell \times r_\ell$

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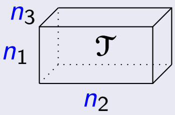
Tensor Unfolding



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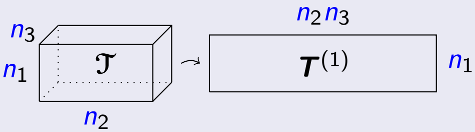

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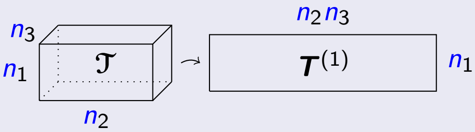

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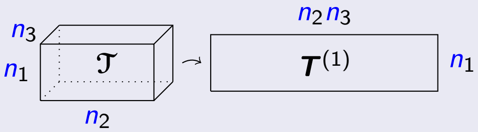

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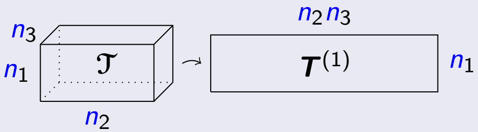

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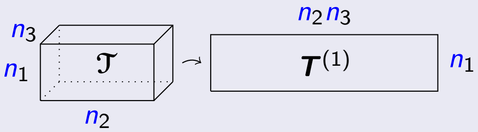

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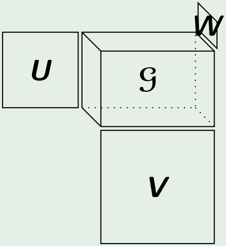

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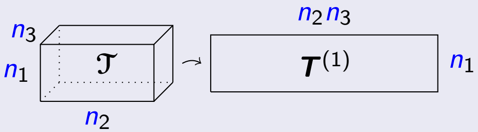

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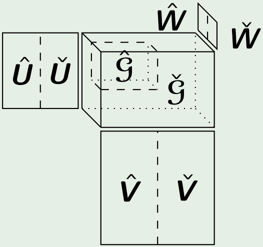

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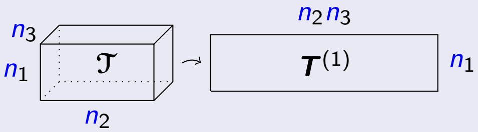

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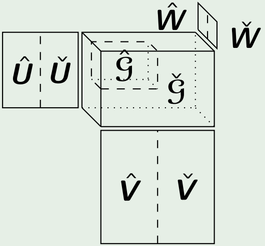

Multilinear Rank

$\hat{\mathcal{T}} = \left[\!\!\left[\hat{\mathcal{G}}; \hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}\right]\!\!\right]$ has *multilinear rank* (r_1, r_2, r_3).

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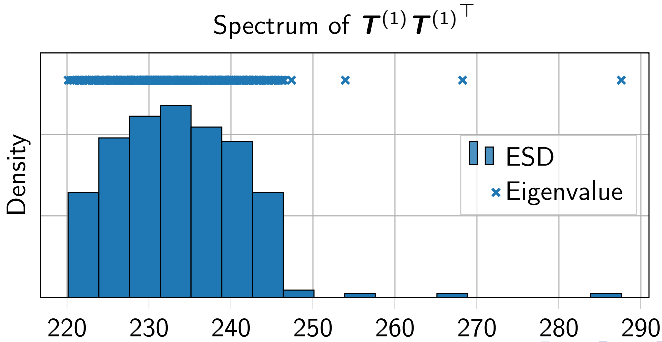

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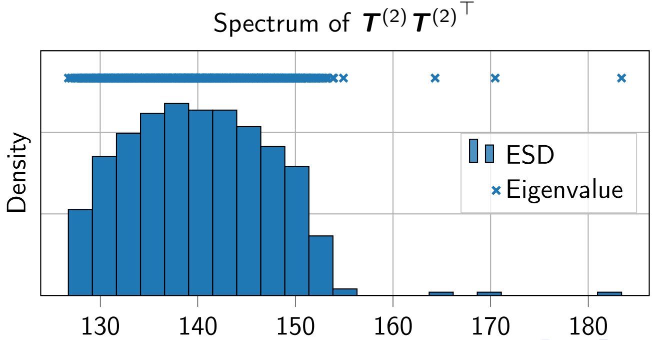

Limiting Spectral Distribution

- $\mathcal{T} = [\![\mathcal{H}; \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]\!] + \frac{1}{\sqrt{N}}\mathcal{N}$.
- $\hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}$ gather the first left singular vectors of $\boldsymbol{T}^{(1)}, \boldsymbol{T}^{(2)}, \boldsymbol{T}^{(3)}$.

➡ We study the limiting spectrum of $\boldsymbol{T}^{(\ell)}\boldsymbol{T}^{(\ell)\top}$.

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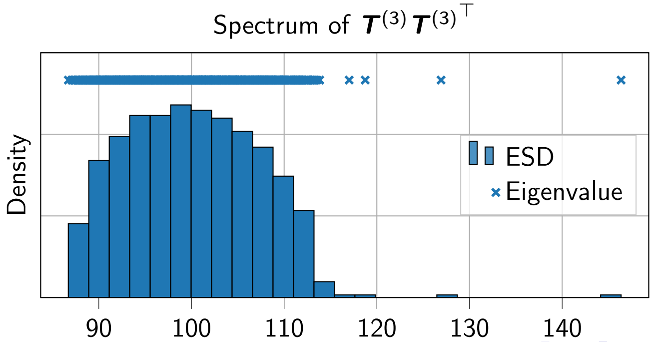

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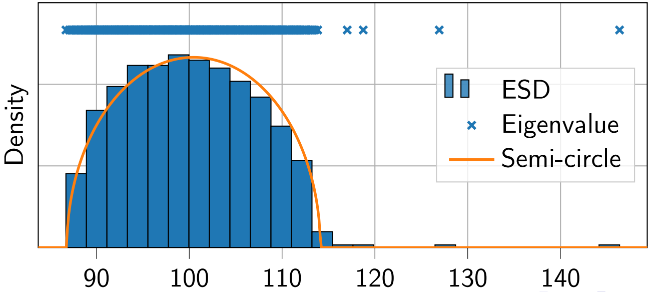
Spectrum of $\boldsymbol{T}^{(3)}\boldsymbol{T}^{(3)\top}$



Limiting Spectral Distribution

- $\mathcal{T} = [\![\mathcal{H}; \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]\!] + \frac{1}{\sqrt{N}}\mathcal{N}$.
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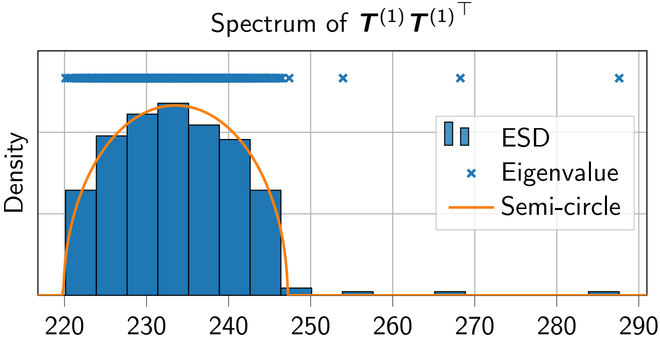
Theorem (Limiting Spectral Distribution)

$$\mu_N^{(\ell)} = \frac{n_1 n_2 n_3}{n_\ell N} = \mathcal{O}(N^{d-2}), \qquad \sigma_N = \frac{\sqrt{n_1 n_2 n_3}}{N} = \mathcal{O}(N^{\frac{d-2}{2}}).$$

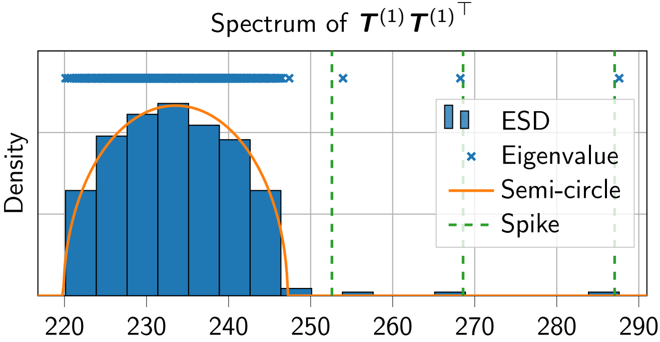
As $N \to +\infty$, $\frac{1}{\sigma_N}\boldsymbol{T}^{(\ell)}\boldsymbol{T}^{(\ell)\top} - \frac{\mu_N^{(\ell)}}{\sigma_N}\boldsymbol{I}_{n_\ell}$ has a limiting spectral distribution $\tilde{\nu}$ whose Stieltjes transform $\tilde{m}$ satisfies

$$\tilde{m}^2(\tilde{z}) + \tilde{z}\tilde{m}(\tilde{z}) + 1 = 0 \qquad \tilde{z} \in \mathbb{C} \setminus \operatorname{supp} \tilde{\nu}$$

Spikes Behavior



Spikes Behavior

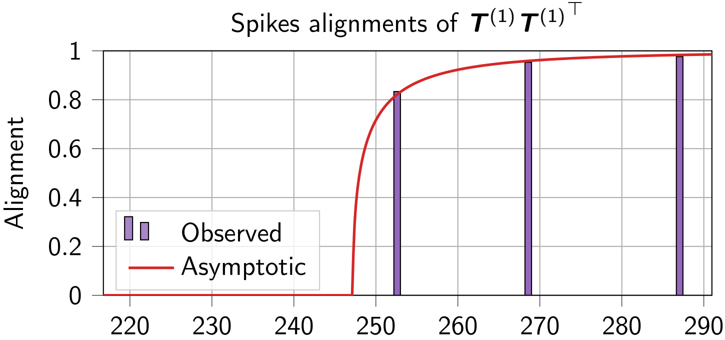


Theorem (Spikes Behavior)

If $\rho_{\kappa,N}^{(\ell)} \equiv \frac{s_\kappa^2(\boldsymbol{P}^{(\ell)})}{\sigma_N} > 1$, *then*

$$s_\kappa^2(\boldsymbol{T}^{(\ell)}) \underset{N\to+\infty}{\overset{a.s.}{\sim}} \rho_{\kappa,N}^{(\ell)} + \frac{1}{\rho_{\kappa,N}^{(\ell)}}, \qquad \left\langle \hat{\boldsymbol{u}}_{\kappa,N}^{(\ell)}, \boldsymbol{x}_\kappa^{(\ell)} \right\rangle^2 \underset{N\to+\infty}{\overset{a.s.}{\sim}} 1 - \frac{1}{\rho_{\kappa,N}^{(\ell)\,2}}.$$

Spikes Behavior



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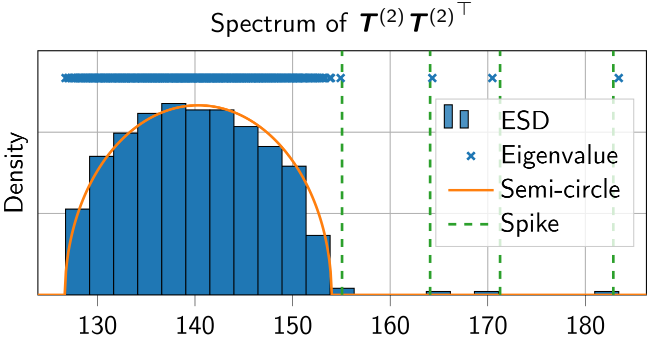

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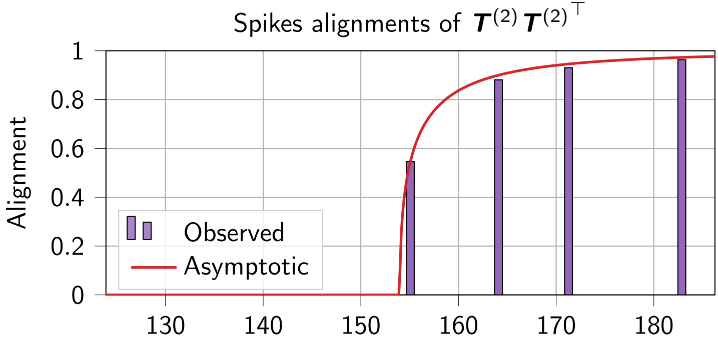
Spikes alignments of $\boldsymbol{T}^{(2)}\boldsymbol{T}^{(2)^\top}$

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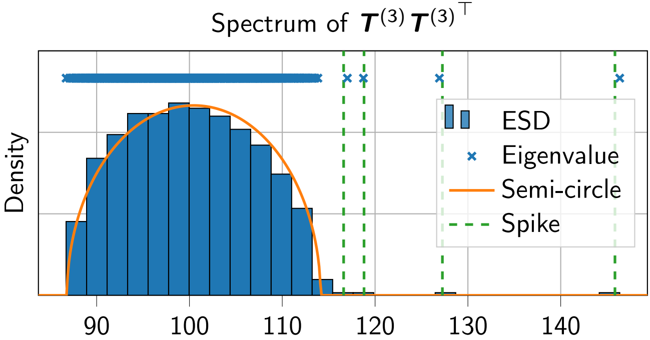
Spikes Behavior



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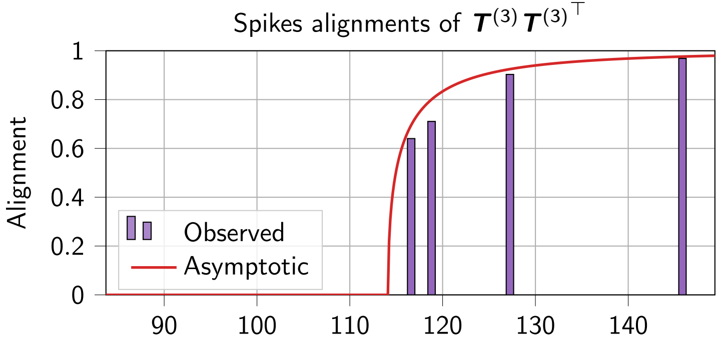
Spikes Behavior



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Conclusion

$$\mathcal{T} = \underbrace{[\![\mathcal{H}; \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]\!]}_{\text{signal (low rank)}} + \underbrace{\frac{1}{\sqrt{N}}\mathcal{N}}_{\text{noise (random)}} \in \mathbb{R}^{n_1 \times n_2 \times n_3},$$

$$N = n_1 + n_2 + n_3$$

$$\mathcal{N}_{i,j,k} \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0, 1)$$

Question

Given $\hat{\mathcal{T}} = \left[\!\!\left[\hat{\mathcal{G}}; \hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}\right]\!\!\right]$, how close are $\hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}$ from $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$?

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Answer — Subspace Alignments

$$\frac{1}{r_1}\left\|\boldsymbol{X}^\top \hat{\boldsymbol{U}}\right\|_{\mathrm{F}}^2$$

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Answer — Subspace Alignments

$$\frac{1}{r_1}\left\|\boldsymbol{X}^\top \hat{\boldsymbol{U}}\right\|_{\mathrm{F}}^2 = \frac{1}{r_1}\sum_{\kappa=1}^{r_1} \cos^2 \theta_{\kappa, N}^{(1)}(\boldsymbol{X}, \hat{\boldsymbol{U}})$$

Conclusion

$$\mathcal{T} = \underbrace{[\![\mathcal{H}; \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}]\!]}_{\text{signal (low rank)}} + \underbrace{\frac{1}{\sqrt{N}}\mathcal{N}}_{\text{noise (random)}} \in \mathbb{R}^{n_1 \times n_2 \times n_3},$$

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κ-th principal angle

Conclusion

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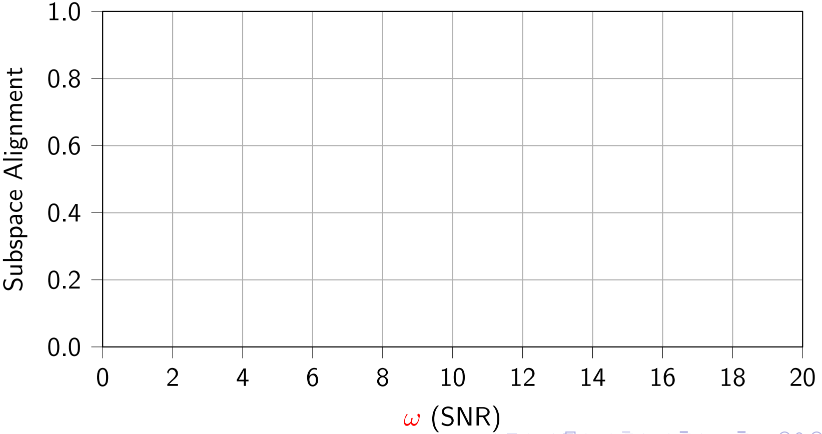
κ-th principal angle

Perspectives

$$\mathfrak{T} = \sqrt{\omega}\mathcal{P}_0 + \frac{1}{\sqrt{N}}\mathcal{N} \qquad \text{with} \qquad \|\mathcal{P}_0\|_{\mathsf{F}}^2 = \sigma_N$$



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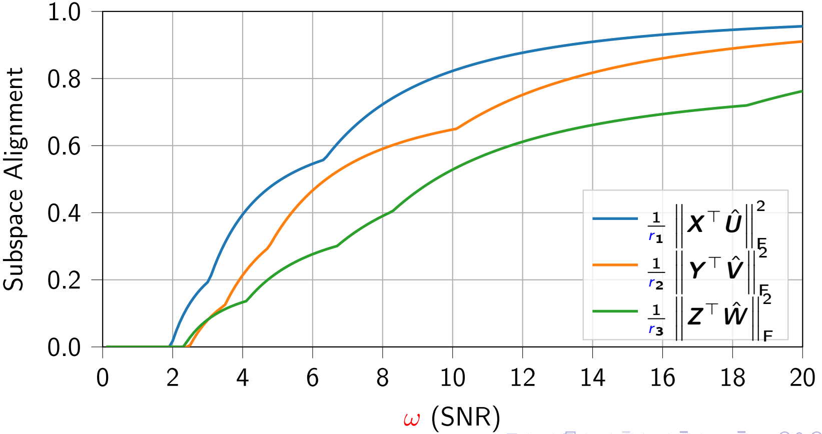
Truncated HOSVD

$$\hat{\mathcal{T}} = \left[\!\left[\hat{\mathcal{G}}; \hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}\right]\!\right]$$



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$$\mathfrak{T} = \sqrt{\omega}\,\mathcal{P}_0 + \frac{1}{\sqrt{N}}\mathcal{N} \qquad \text{with} \qquad \|\mathcal{P}_0\|_{\mathrm{F}}^2 = \sigma_N$$

Truncated HOSVD

$$\hat{\mathfrak{T}} = [\![\hat{\mathcal{G}}; \hat{\boldsymbol{U}}, \hat{\boldsymbol{V}}, \hat{\boldsymbol{W}}]\!]$$

Maximum Likelihood

$$\mathfrak{T}_\star = [\![\mathcal{G}_\star; \boldsymbol{U}_\star, \boldsymbol{V}_\star, \boldsymbol{W}_\star]\!]$$