Tensor Estimation at Growing Rank

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Random Tensors

Symmetric Tensor Estimation

Let $\mathbf{X}_0$ be an $N \times M$ signal matrix with i.i.d. entries drawn from some centred, bounded distribution $\mathbb{P}_X$ and let $\mathbf{Z}$ be an order p tensor with i.i.d. $\mathcal{N}(0,1)$ entries. Consider the order p output

$$\mathbf{Y} = \sqrt{\frac{\lambda(p-1)!}{N^{p-1}}} \sum_{k=1}^{M} \mathbf{X}_{0,\cdot,k}^{\otimes p} + \mathbf{Z},$$

$$\mathbf{Y}_{i_1,\dots,i_p} = \sqrt{\frac{\lambda(p-1)!}{N^{p-1}}} \sum_{k=1}^{M} \mathbf{X}_{0,i_1,k}\mathbf{X}_{0,i_2,k} \cdots \mathbf{X}_{0,i_p,k} + \mathbf{Z}_{i_1,\dots,i_p}.$$

with SNR λ.

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Goal: Understand the mutual information $I(\mathbf{X}_0; \mathbf{Y})$ at large N in the Bayes-optimal setting.

Symmetric Tensor Estimation

Bayes-optimal means that we may assume that the estimator $\mathbf{X}$ of $\mathbf{X}_0$ is such that

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The posterior distribution is

$$\mathbb{P}(\mathbf{X}|\mathbf{Y}) = \frac{\mathbb{P}_X^{\otimes MN}}{\mathcal{Z}_N(\mathbf{Y})} e^{\mathcal{H}_N(\mathbf{X})},$$

$$\mathcal{H}_N(\mathbf{X}) = -\frac{1}{2} \sum_{i_1 \leq i_2 \leq \cdots \leq i_p} \Bigg(\mathbf{Y}_{i_1,\ldots,i_p} - \sqrt{\frac{\lambda(p-1)!}{N^{p-1}}} \times \sum_{k=1}^{M} \mathbf{X}_{i_1,k} \mathbf{X}_{i_2,k} \cdots \mathbf{X}_{i_p,k} \Bigg)^2 + \frac{1}{2} \mathbf{Y}_{i_1,\ldots,i_p}^2$$

Symmetric Tensor Estimation

Letting $F_N(\lambda) = \frac{1}{NM}\mathbb{E}_{\mathbf{Z},\mathbf{X}_0} \ln \mathcal{Z}_N(\mathbf{Y})$, we have that

$$\lim_{N\to\infty} \frac{1}{N} I(\mathbf{X}_0; \mathbf{Y}) = \frac{\lambda}{2p} \sum_{k,k'=1}^{M} (\mathbb{E}[\mathbf{X}_{0,1}\mathbf{X}_{0,1}^T]^{\circ p})_{kk'} - \lim_{N\to\infty} F_N(\lambda).$$

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For finite M it's known [Luneau-Barbier-Macris,'21] that

$$\lim_{N\to\infty} F_N(\lambda) = \sup_{\mathbf{Q}\in\mathcal{S}_M^+} F_{M,p}^{\mathrm{RS}}(\mathbf{Q},\lambda),$$

$$F_{M,p}^{\mathrm{RS}}(\mathbf{Q},\lambda) = \frac{1}{M}\mathbb{E} \ln \int e^{\sqrt{\lambda}\mathbf{x}^T\sqrt{\mathbf{Q}^{\circ(p-1)}}\mathbf{z} + \lambda \mathbf{x}_0^T \mathbf{Q}^{\circ(p-1)}\mathbf{x} - \frac{\lambda}{2}\mathbf{x}^T\mathbf{Q}^{\circ(p-1)}\mathbf{x}}\, d\mathbb{P}_X^{\otimes M}(\mathbf{x})$$

$$- \frac{\lambda(p-1)}{2pM} \sum_{k,k'=1}^{M} (\mathbf{Q}^{\circ p})_{kk'}.$$

Matrix Estimation at Growing Rank

Theorem (Barbier-Ko-R., '24)

Setting $p = 2$ and assuming that $M = \mathrm{o}(N^{1/10})$, we have the Parisi-type formula

$$\lim_{N\to\infty} F_N(\lambda) = \sup_{q\in[0,\rho]} F^{\mathrm{RS}}_{1,2}(q,\lambda).$$

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Key ideas:

- $\sup F^{\mathrm{RS}}_{M,2} = \sup F^{\mathrm{RS}}_{1,2}$
- Overlap concentration
- A multiscale cavity method decoupling M, N growth

Extending to Tensors

We proved the p=2 rank-one reduction

$$\sup_{\mathbf{Q}\in\mathcal{S}_M^+} F_{M,2}^{\mathrm{RS}}(\mathbf{Q},\lambda) = \sup_{q\in[0,\rho]} F_{1,2}^{\mathrm{RS}}(q,\lambda)$$

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In the general-p case, we need to relate eigenvalues of $\mathbf{Q}^{\circ(p-1)}$ to those of $\mathbf{Q}^{\circ p}$ or find some other way of maximising

$$F^{\mathrm{RS}}_{M,p}(\mathbf{Q},\lambda) = \frac{1}{M}\mathbb{E}\ln\int e^{\sqrt{\lambda}\mathbf{x}^T\sqrt{\mathbf{Q}^{\circ(p-1)}}\mathbf{z}+\lambda\mathbf{x}_0^T\mathbf{Q}^{\circ(p-1)}\mathbf{x}-\frac{\lambda}{2}\mathbf{x}^T\mathbf{Q}^{\circ(p-1)}\mathbf{x}}\,d\mathbb{P}_X^{\otimes M}(\mathbf{x})$$

$$-\frac{\lambda(p-1)}{2pM}\sum_{k,k'=1}^{M}(\mathbf{Q}^{\circ p})_{kk'}.$$

Questions?

References: arXiv:2403.07189