Regular language quantum states

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Random tensors and related topics - 02.10.2024

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Motivation: limitations of the MPS framework

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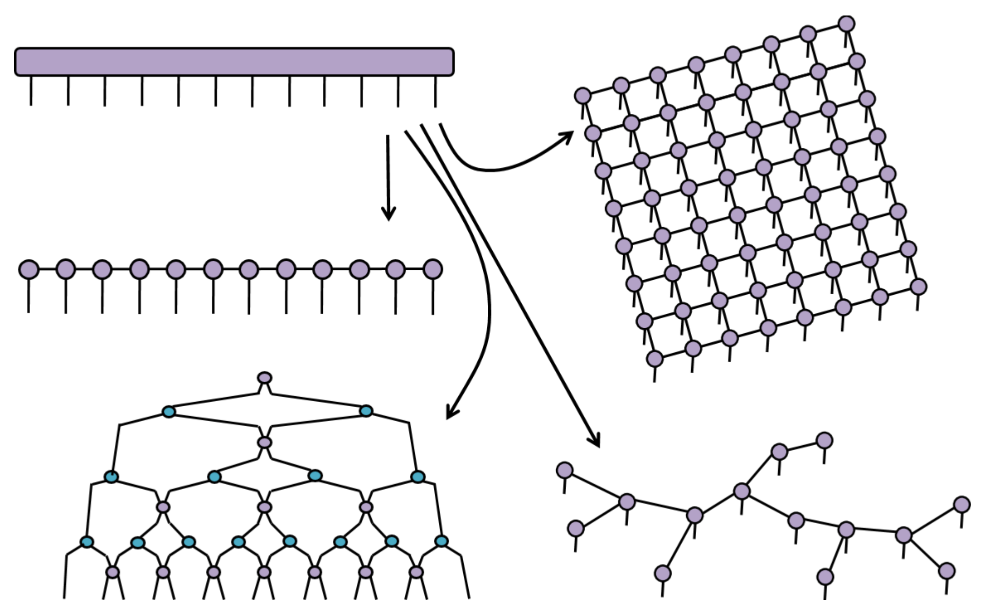
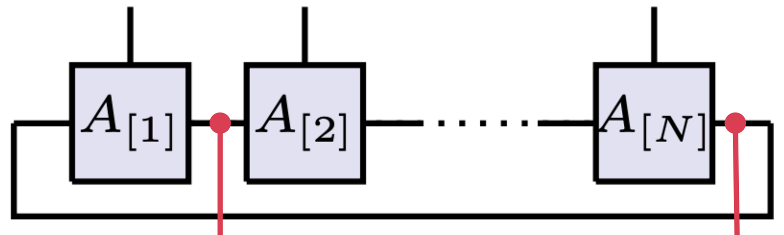
Tensor networks (TNs)



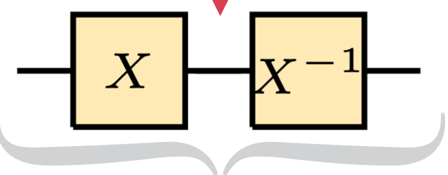
1D: Matrix Product States (MPS)

$$|\psi\rangle := \sum_{i_1\dots i_N=1}^{d} \mathrm{Tr}\left[A_{[1]}^{i_1} A_{[2]}^{i_2} \dots A_{[N]}^{i_N}\right] |i_1 \dots i_N\rangle$$

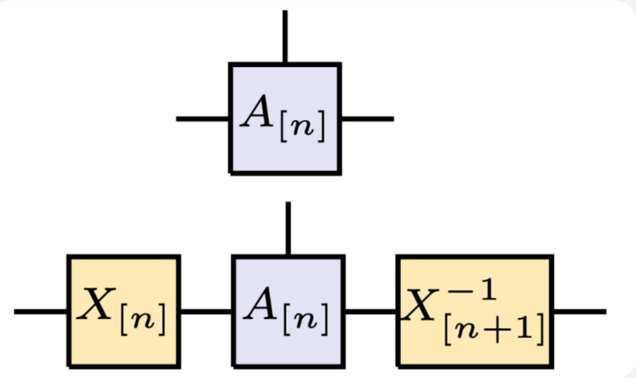
$=$ 

$\in (\mathbb{C}^d)^{\otimes N},$

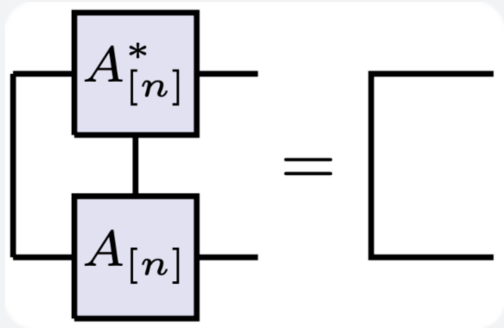
Bond dimension



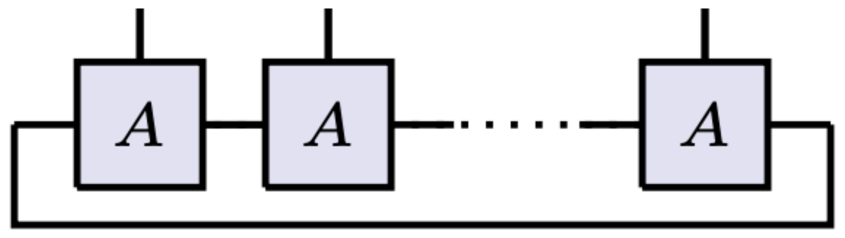
They efficiently and faithfully represent low-energy states of local Hamiltonians.



Freedom in the representation Convenient forms can be imposed:



Uniform matrix product states:



Translationally-invariant & Periodic boundary conditions

Motivation: limitations of the MPS framework

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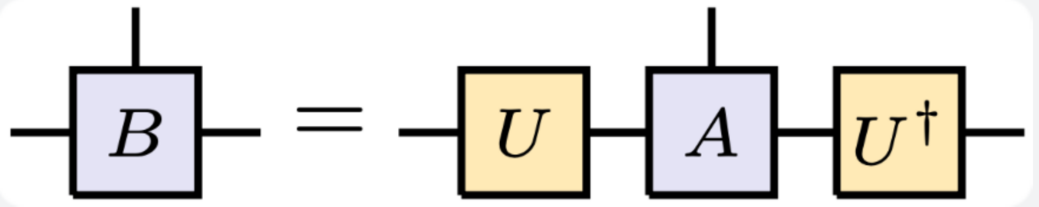
- **Canonical form** in terms of basic building blocks (basis of normal tensors): $A^i = \bigoplus_{j=1}^{b} \bigoplus_{q=1}^{r_j} \mu_{j,q} A^i_j$

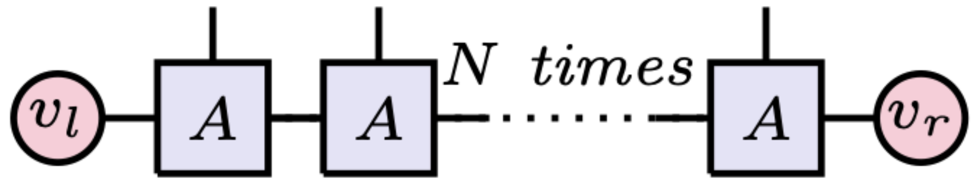
- **Fundamental theorem:** Two tensors *A*, *B* in canonical form generate the same state for all *N* if and only if they are related as:

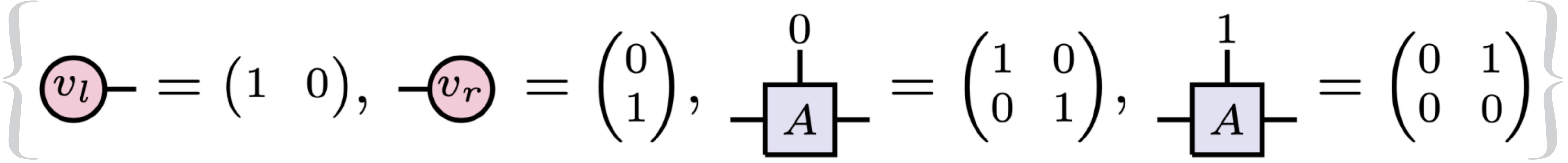


➡ Classification of topological phases in the MPS framework and their symmetry-enriched counterpart.

- **But this framework is not always valid:** There are translationally invariant states that are MPS but do not admit a uniform MPS representation!

$|W_N\rangle \propto |10\ldots0\rangle + \ldots + |0\ldots01\rangle =$





- Any uniform MPS representation of it has a bond dimension scaling with the system size ($D \geq \Omega(N^{1/(3+\delta)})$).

Regular language states: regular expressions

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- Alphabet $\Sigma := \{0,\ldots,d-1\}$
- Word (any finite sequence of letters of Σ)
- Σ^* (all possible words of arbitrary length)

➡ **Language:** collection of words of Σ^*

Regular expression:

- $\varnothing$
- $a \in \Sigma$ (single letter)
- ε (empty character)
- R_1R_2 (concatenation)
- $R_1 \cup R_2$ (union)
- $R_1^* = \varepsilon \cup R_1 \cup R_1R_1 \cup \ldots$ (Kleene star)

Regular languages

Examples:
($\Sigma := \{0,1\}$)

- $0^* \cup 1^*$ → $|00\ldots0\rangle + |11\ldots1\rangle$ (GHZ state)
- 0^*10^* → $|10\ldots0\rangle + |01\ldots0\rangle + \ldots + |00\ldots1\rangle$ (W state)
- $0^*10^*10^*$ → $|110\ldots0\rangle + |101\ldots0\rangle + \ldots |00\ldots11\rangle$ (Dicke state)
- 0^*12^* → $\sum |0\ldots012\ldots2\rangle$ (simplified domain wall)
- $((0^*12^*) \cup (4^*0)^*)^*3(5^*21)^*, \ldots$

Regular language states (RLS):
Family of quantum states associated to a RL, L, is $L_q = \{|L_N\rangle\}_{N\in\mathbb{N}}$ where

$$|L_N\rangle = \sum_{w\in L\cap\Sigma^N} |w\rangle$$

Regular language states: finite automata

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NFA ((non-deterministic) finite automaton):

$\mathscr{F} = \langle Q, \Sigma, \delta, I, F \rangle$

- Q (set of internal states)
- Σ (alphabet)
- δ (transition function)
- I (set of initial states)
- F (set of accepting states)

A word $w = x_1 x_2 \ldots x_n$ is **accepted** by $\mathscr{F}$ if there is at least one path along the NFA:

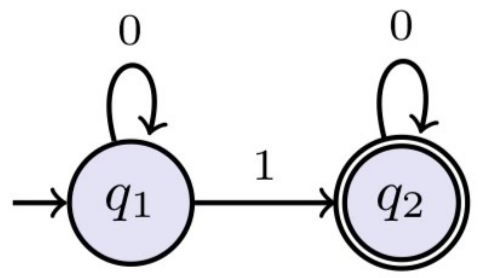
$$r_0 \xrightarrow{x_1} r_1 \xrightarrow{x_2} r_2 \xrightarrow{x_3} \ldots \xrightarrow{x_n} r_n \quad \begin{cases} r_0 \in I \\ r_n \in F \\ r_{i+1} \in \delta(r_i, x_{i+1}) \end{cases}$$

Kleene's theorem

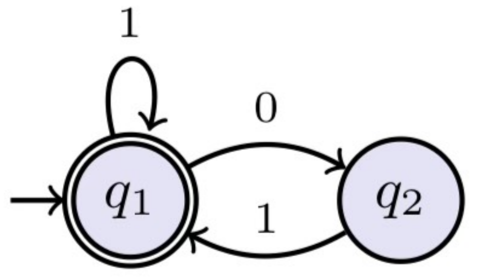
L is described by a regular expression
$\Longleftrightarrow$
L is accepted by a finite automaton

Examples:

$L_1 = 0^*10^*$ ➡ $\mathcal{F}_1$:



$L_2 = 1^*(011^*)^*$ ➡ $\mathcal{F}_2$:



NFA as matrix product states:

Define:

$$\begin{cases} (v_l)- := \sum_{i \in I} \langle i | , \\ -(v_r) := \sum_{f \in F} | f \rangle , \end{cases} \qquad i - \overset{x}{\boxed{A}} - j = \begin{cases} 1 \ \textit{if} \ j \in \delta(i, x), \\ 0 \ \textit{otherwise}. \end{cases}$$

$$| L_N \rangle := (v_l) - \boxed{A} - \boxed{A} - \overset{N \ times}{\cdots\cdots} - \boxed{A} - (v_r) = \sum_{w \in L \cap \Sigma^N} \boxed{c_w} | w \rangle$$

number of accepting paths for w

If the NFA is **unambiguous** (UFA) $\Longrightarrow$ The MPS is a RLS.

Only one path per word (i.e. $c_w = 1, \; \forall w$) ⟶ A UFA accepting a RL L always exists.

What can this **connection** be useful for?

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Tensor networks → Regular languages

Using MPS tools one can check $\left\{\begin{array}{l}\text{if a MPS is a RLS (i.e. if a NFA is unambiguous)} \\ \text{if a regular language is shift-invariant}\end{array}\right\}$ more efficiently.

Regular languages → Tensor networks

Is there a canonical choice for the tensors? ➡ Automata theory provides a canonical minimal determinstic finite automaton (DFA) that can be **efficiently** found.

- ➡ **Minimal** (among all the RL **deterministic** representations)
- ➡ **Unique** (up to relabelling of the internal states)

Can we address physically relevant questions with it?

➡ When are two RLS equivalent under **local unitary (LU)** operations? ➡ Is there a unitary U s.t. $|L_2^N\rangle = U^{\otimes N}|L_1^N\rangle\,, \ \forall N$?

✓ Identify alphabet symbols: $\Sigma = \Sigma_\infty \cup \boxed{\Sigma_f}$ → (number of appearances of Σ_f symbols in any word is upper bounded by a constant M)

✓ A **canonical decomposition** of the RLS can be obtained: $|L^N\rangle = \sum_m \sum_j \hat{S}^{(m)} \underbrace{\boxed{|L_j^m\rangle}}_{\Sigma_\infty} \underbrace{\boxed{|X_j^m\rangle}}_{\Sigma_f}$

✓ For $\underbrace{\text{sparse RLS}}_{\||L_N\rangle\| = O(\mathrm{poly}(N))}$: **fundamental theorem** for LU equivalence: $|L_2^N\rangle = U^{\otimes N}|L_1^N\rangle\,,\ \forall N \iff \underbrace{\left\{\begin{array}{l} L_{2,j}^m = \pi(L_{1,j}^m), \\ |X_{2,j}^{(m)}\rangle := U_f^{\otimes m}|X_{1,j}^m\rangle \end{array}\right.}_{\text{(finite number of manageable subproblems)}}$

Outlook

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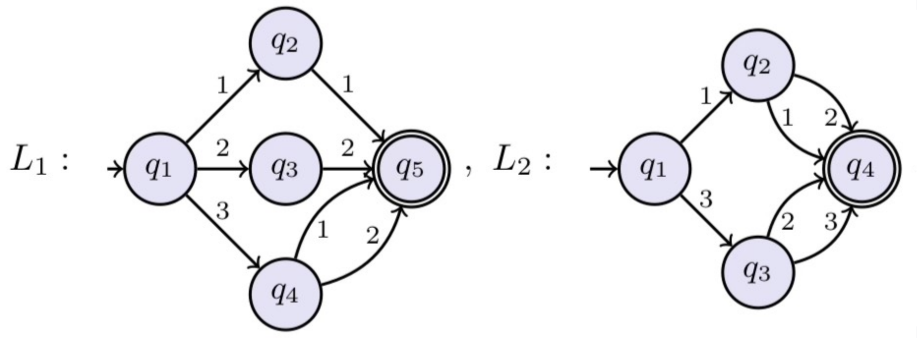
- Open questions:

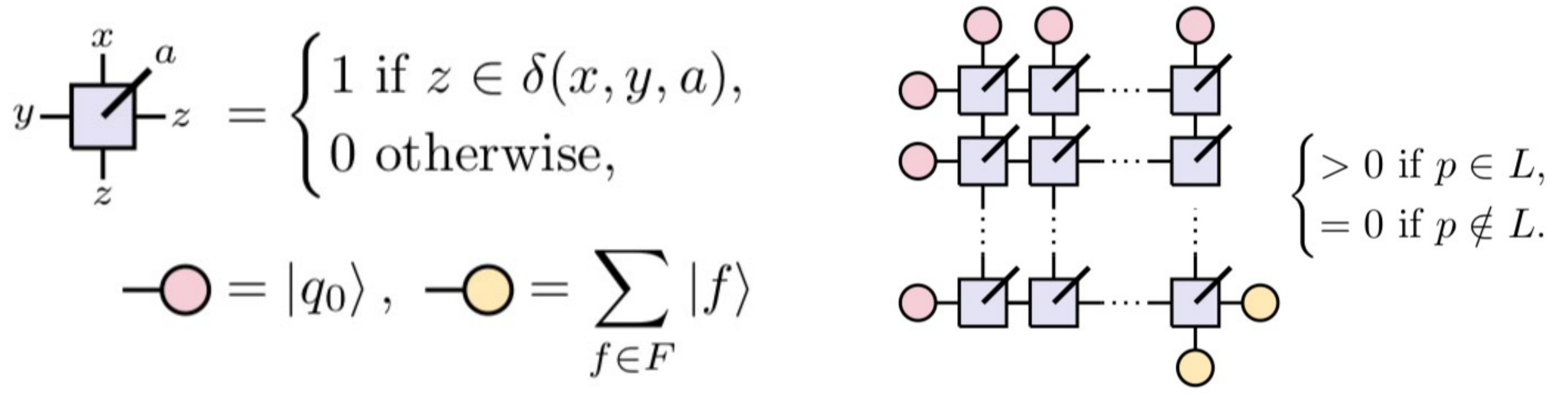
The MPS bond dimension does not change under LU operations. $\Longleftrightarrow$ The state complexity can change under LU operations. ➡ Interplay of entanglement and RL complexity measures?

$$\begin{cases} L_1 = 11 \cup 22 \cup 31 \cup 32, \\ L_2 = 11 \cup 12 \cup 32 \cup 33, \end{cases}$$

$$|L_2\rangle = U^{\otimes 2} |L_1\rangle$$



- Generalizations to broader classes of formal languages:
  - Online tessellation automata in 2D (OTA)
  - Pushdown automata



Regular languages ⇄ **Tensor networks**