ENTANGLEMENT BETWEEN CONES

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1. Lecture 1

We work in a finite-dimensional real vector space V.

1.1. **Cones.** A subset $\mathsf{C} \subset V$ is a (convex) cone if $sx + ty \in \mathsf{C}$ whenever $s, t \in \mathbf{R}^+$ and $x, y \in \mathsf{C}$. We usually assume some conditions on C. We say that C is

- *generating* if C spans V as a vector space, or equivalently is not contained in a proper subspace of V, or again if $\mathsf{C} - \mathsf{C} = \{x - y \ : \ x, y \in \mathsf{C}\} = V$.
- *salient* if C does not contain a line, or equivalently if 0 is an extreme point of C, or again if $\mathsf{C} \cap (-\mathsf{C}) = \{0\}$.
- *proper* if C is closed, generating and salient.

Two cones $\mathsf{C}_1 \subset V_1$ and $\mathsf{C}_2 \subset V_2$ are isomorphic if there exist a linear bijection $\Phi : V_1 \to V_2$ such that $\Phi(\mathsf{C}_1) = \mathsf{C}_2$.

Here are some example of cones.

(1) The *positive orthant* $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n \ : \ x_i \geqslant 0\}$ is a proper cone in the vector space $\mathbf{R}^n$. A cone isomorphic to $\mathbf{R}^n_+$ is called classical. Equivalently, C is classical if there is a basis $(e_1, \dots, e_n)$ of the vector space V which generates C, is the sense that
$$\mathsf{C} = \left\{ \sum_{i=1}^n \lambda_i e_i \ : \ \lambda_i \geqslant 0 \right\}.$$
(2) The *Lorentz cone* $\mathsf{L}_n = \{x \in \mathbf{R}^n \ : \ x_n \geqslant (x_1^2 + \cdots + x_{n-1}^2)^{1/2}\}$.
(3) The cone of positive semi-definite operators over a real or complex Hilbert space $\mathsf{PSD}(H)$ is a proper cone in the vector space $\mathrm{Herm}(H)$ of Hermitian operators. If $\dim(H) = n$ then the dimension of $\mathsf{PSD}(H)$ equals $n(n+1)/2$ in the real case and n^2 in the complex case.
(4) If X is a finite-dimensional normed space, the cone over X is the subset of $X \times \mathbf{R}$ defined as
$$\mathsf{C}_X = \{(x, t) \ : \ t \geqslant \|x\|\}$$

Exercise 1. All two-dimensional proper cones are classical. The cones $\mathsf{PSD}(\mathbf{R}^2)$ and L_3 are isomorphic. The cones $\mathsf{PSD}(\mathbf{C}^2)$ and L_4 are isomorphic.

1.2. **Extreme rays.** A nonzero vector $x \in \mathsf{C}$ is extremal if, whenever $x = x' + x''$ for x', x'' in C, then x' and x'' are proportional to x. If x is extremal, the set $\{\lambda x \ : \ \lambda \geqslant 0\}$ is called an extreme ray of C. Theorem: any proper cone is the convex hull of its extreme rays.

Exercise 2. Show that any d-dimensional proper cone has $\geqslant d$ extreme rays, and that any proper cone with d extreme rays is classical. Show that all 3-dimensional proper cones with 4 extreme rays are isomorphic.

1.3. **Duality.** Let $\mathsf{C} \subset V$ be a cone, and V^* be the dual vector space. We define the dual cone as

$$\mathsf{C}^* = \{f \in V^* \ : \ f(x) \geqslant 0 \text{ for every } x \in \mathsf{C}\}.$$

Since V is finite-dimensional, we may identify V^{**} with V. Then

Proposition 1. *If $\mathsf{C} \subset V$ is a nonempty closed convex cone, then $\mathsf{C}^{**} = \mathsf{C}$.*

Proof. A point x belongs to C^{**} iff $f(x) \geqslant 0$ for every $f \in \mathsf{C}^*$, so the inclusion $\mathsf{C} \subset \mathsf{C}^{**}$ is obvious. If $x \notin \mathsf{C}$, we may find by the Hahn–Banach separation theorem (since C is closed and convex) a linear form $f \in V^*$ such that $f(x) < \inf f(\mathsf{C})$. Since $f(\mathsf{C})$ is a nonempty cone in $\mathbf{R}$, since is only possible if that infimum equals 0. This means that $f \in \mathsf{C}^*$ and therefore $x \notin \mathsf{C}^{**}$. □

Duality reverses order: if $\mathsf{C}_1 \subset \mathsf{C}_2$ then $\mathsf{C}_1^* \supset \mathsf{C}_2^*$. If W is a linear subspace (a very special kind of closed convex cone), then W^* is the annihilator of W. In particular, it follows that, for a closed convex cone C, the cone C is generating iff the dual cone C^* is salient.

Proposition 2. *Let $\mathsf{C} \subset V$ be a closed convex cone. A linear form $f \in V^*$ belongs to the interior of C^* iff we have $f(x) > 0$ for every $x \in \mathsf{C} \backslash \{0\}$.*

Proof. Consider an arbitrary norm $\|\cdot\|$ on V and let $\|\cdot\|_*$ be the dual norm on V^*. The key is to observe that, for $f \in V^*$

$$\inf_{\|h\|_* \leqslant 1} (f + \varepsilon h)(x) = f(x) - \varepsilon \|x\|.$$

and therefore $B(f, \varepsilon) \subset \mathsf{C}^*$ iff $f(x) \geqslant \varepsilon \|x\|$ for every $x \in \mathsf{C}$. By homogeneity, this is equivalent to say that f is positive on the compact set $\mathsf{C} \cap S$ (where S is the unit sphere). □

The dual version is: $x \in \operatorname{int}(\mathsf{C})$ iff $f(x) > 0$ for any nonzero $f \in \mathsf{C}^*$. In particular, if $x \in \partial \mathsf{C}$, there exists a nonzero $f \in \mathsf{C}^*$ such that $f(x) = 0$.

A base for a cone $\mathsf{C} \subset V$ is a convex set $B \subset \mathsf{C}$ such that the map $(t, x) \mapsto tx$ is a bijection from $(0, \infty) \times B$ to C.

Proposition 3. *Let C be a proper cone. For any $f \in \operatorname{int}(\mathsf{C}^*)$, the set $\{x \in \mathsf{C} \ : \ f(x) = 1\}$ is a base for C, which is compact.*

Proof. Take $f \in \operatorname{int}(\mathsf{C}^*)$. Any $x \in \mathsf{C} \backslash \{0\}$ can be written as $\lambda \cdot x / \lambda$ for $\lambda = f(x)$, hence $\mathsf{C} \cap \{f = 1\}$ is a base. Use local compactness of the ambiant space to conclude. □

If K_1 and K_2 are two different bases for C, there is a projective transformation $p : K_1 \to K_2$ mapping K_1 to K_2.

Examples (1)-(3) of cones given earlier have a stronger property: any different bases are related by an affine transformation. This is related to the fact that the cones are homogeneous (a cone C is homogeneous if the group of automorphisms act transitively on $\operatorname{int}(\mathsf{C})$), which is very special situation.

If V is equipped with an inner product, any element of V^* can be realized as $x \mapsto \langle x, y \rangle$ for a unique $y \in V$. This allows to identify V and V^*. This is the case for the space $\mathbf{R}^n$ (using the usual inner product) and for the space $\operatorname{Herm}(H)$, using the Hilbert–Schmidt inner product $\langle A, B \rangle = \operatorname{Tr}(AB)$.

Exercise 3. Show that with the above identification, we have $(\mathbf{R}_+^n)^* = \mathbf{R}_+^n$, $\mathsf{L}_n^* = \mathsf{L}_n$ and $\mathsf{PSD}(H)^* = \mathsf{PSD}(H)$.

Show that for every normed space X, the cone $(\mathsf{C}_X)^*$ is isomorphic to C_{X^*}.

1.4. **Tensor products of cones.** Suppose now that we have two proper cones $\mathsf{C}_1 \subset V_1$ and $\mathsf{C}_2 \subset V_2$ and we want to define $\mathsf{C}_1 \otimes \mathsf{C}_2$. There are two meaningful definitions.

The *minimal tensor product* of C_1 and C_2 is defined as

$$\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2 = \operatorname{conv}\{x_1 \otimes x_2 \ : \ x_1 \in \mathsf{C}_1,\ x_2 \in \mathsf{C}_2\}.$$

Example 1. Given Hilbert spaces H_1 and H_2, there is a natural embedding

$$\operatorname{Herm}(H_1) \otimes \operatorname{Herm}(H_2) \longrightarrow \operatorname{Herm}(H_1 \otimes H_2)$$

This embedding is surjective in the complex case but not in the real case (compare dimensions). In the complex case, we identify $\operatorname{Herm}(H_1) \otimes \operatorname{Herm}(H_2)$ and $\operatorname{Herm}(H_1 \otimes H_2)$. The cone $\mathsf{PSD}(H_1) \otimes_{\min} \mathsf{PSD}(H_2)$ is the cone of separable operators. By homogeneity, we may restrict to states (positive operators with trace 1). A state is separable iff it is a convex combination of pure product states.

Exercise 4. If C_1 and C_2 are proper, so is $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$.

There is a dual notion, the *maximal tensor product* of C_1 and C_2 defined as

$$\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2 = \{z \in V_1 \otimes V_2 \ : \ (f_1 \otimes f_2)(z) \geqslant 0 \text{ for every } f_1 \in \mathsf{C}_1^*,\ f_2 \in \mathsf{C}_2^*\},$$

or more succinctly

$$\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2 = (\mathsf{C}_1^* \otimes_{\min} \mathsf{C}_2^*)^*.$$

Example 2. Given Hilbert spaces H_1 and H_2, $\mathsf{PSD}(H_1) \otimes_{\max} \mathsf{PSD}(H_2)$ identifies with the cone of block-positive operators. An operator $T \in \operatorname{Herm}(H_1 \otimes H_2)$ is said to be block-positive if

$$\langle x_1 \otimes x_2 | T | x_1 \otimes x_2 \rangle \geqslant 0$$

for every $x_1 \in H_1$ and $x_2 \in H_2$. They are sometimes called entanglement witnesses in quantum information. We have strict inclusions

$$\mathsf{PSD}(\mathsf{C}^2) \otimes_{\min} \mathsf{PSD}(\mathsf{C}^2) \subsetneq \mathsf{PSD}(\mathsf{C}^2 \otimes \mathsf{C}^2) \subsetneq \mathsf{PSD}(\mathsf{C}^2) \otimes_{\max} \mathsf{PSD}(\mathsf{C}^2).$$

To show that the left inclusion is strict, consider the entangled state $|\psi\rangle\langle\psi|$ with $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. To show that the right inclusion is strict, consider the operator $\operatorname{Id} - 2|\psi\rangle\langle\psi|$ which is block-positive but not positive.

For general convex cones C_1 and C_2, it is obvious that

$$\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2 \subset \mathsf{C}_1 \otimes_{\max} \mathsf{C}_2.$$

Let's say that the pair $(\mathsf{C}_1, \mathsf{C}_2)$ is nuclear if $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2 = \mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$, and entangleable if $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2 \subsetneq \mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$.

The goal of these lectures is to prove the following result, confirming a conjecture by Barker in the late 1970's.

Theorem 1 (Aubrun–Lami–Plávala–Palazuelos)**.** *Let $\mathsf{C}_1 \subset V_1$ and $\mathsf{C}_2 \subset V_2$ be two proper cones. Then the pair $(\mathsf{C}_1, \mathsf{C}_2)$ is nuclear if and only if C_1 or C_2 is classical.*

2. Lecture 2

Consider now the cone $\mathsf{P}(\mathsf{C}_1, \mathsf{C}_2)$ of positive operators from C_1 to C_2 (i.e., linear maps $\Phi \in L(V_1, V_2)$ such that $\Phi(\mathsf{C}_1) \subset \mathsf{C}_2$). The space $L(V_1, V_2)$ is canonically isomorphic to $V_1^* \otimes V_2$. Under this isomorphism, the cone $\mathsf{P}(\mathsf{C}_1, \mathsf{C}_2)$ corresponds to $\mathsf{C}_1^* \otimes_{\max} \mathsf{C}_2$. What corresponds to $\mathsf{C}_1^* \otimes_{\min} \mathsf{C}_2$ is the cone of entanglement-breaking or "mention measure-and-prepare" maps, of the form

$$\phi(x) = \sum_i f_i(x) x_i$$

with $f_i \in \mathsf{C}_1^*$ and $x_i \in \mathsf{C}_2$.

One direction is easy: assuming that C_1 is classical, we show that $\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2 \subset \mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$. Let (e_i) be a basis of the vector space V_1 and (e_i^*) be the dual basis. Observe that $\varepsilon_i^* \in \mathsf{C}^*$. We may write z as

$$z = \sum_i e_i \otimes y_i$$

for (y_i) in V_2. For any $f \in \mathsf{C}_2^*$, we have

$$0 \leqslant (e_i^* \otimes f)(z) = f(y_i).$$

Since this holds for every $f \in \mathsf{C}_2^*$, we conclude that $y_i \in \mathsf{C}_2^{**} = \mathsf{C}_2$. This shows that $z \in \mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$.

Given two non-classical cones C_1 and C_2, we need to find an "entangled" vector in $\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$ but not in $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$. We first observe that such a vector cannot be of rank 2.

Proposition 4 (Cariello)**.** *Let $\mathsf{C}_1 \subset V_1$ and $\mathsf{C}_2 \subset V_2$ be proper cones. If $z \in \mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$ has rank $\leqslant 2$, then $z \in \mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$.*

Proof. If $z = x_1 \otimes x_2$, then for every $f \in \mathsf{C}_2^*$ we have $f(x_2)x_1 \in \mathsf{C}_1$, so elements C_2^* have a constant sign on x_2. We may assume that this sign is positive, so that $x_2 \in \mathsf{C}_2$ and $x_1 \in \mathsf{C}_1$.

Assume that z has rank 2 and that $z \in \operatorname{int}(\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2)$. Write $z = x_1 \otimes x_2 + y_1 \otimes y_2$ and let $E_i = \operatorname{span}(x_i, y_i)$. The space E_i intersects the interior of C_i. The 2-dimensional proper cone $\mathsf{C}_i \cap E_i$ has two generators s_i and t_i. Since s_i and t_i belong to the boundary of C_i, by the Hahn–Banach theorem there exists nonzero s_i^* and t_i^* in C_i^* such that $s_i^*(s_i) = t_i^*(t_i) = 0$. We have $s_i^*(t_i) > 0$ and $t_i^*(s_i) > 0$ (because $s_i + t_i \in \operatorname{int}(\mathsf{C}_i)$) and we may rescale such that $s_i^*(t_i) = t_i^*(s_i) = 1$. The map

$$\Phi_i : x \mapsto s_i^*(x) t_i + t_i^*(x) s_i$$

fixes s_i and t_i and hence is the identity on E_i. Hence,

$$\begin{aligned} z &= (\Phi_1 \otimes \Phi_2)(z) \\ &= (s_1^* \otimes s_2^*)(z) t_1 \otimes t_2 + (s_1^* \otimes t_2^*)(z) t_1 \otimes s_2 \\ &\quad + (t_1^* \otimes s_2^*)(z) s_1 \otimes t_2 + (t_1^* \otimes t_2^*)(z) s_1 \otimes s_2 \end{aligned}$$

showing that $z \in \mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$.

For the general case, fix $u_i \in \operatorname{int}(\mathsf{C}_i)$ and $u_i^* \in \operatorname{int}(\mathsf{C}_i^*)$. The vector

$$z_\varepsilon = (\mathrm{Id} + \varepsilon |u_1\rangle\langle u_1^*|) \otimes (\mathrm{Id} + \varepsilon |u_2\rangle\langle u_2^*|)(z)$$

belongs to $\operatorname{int}(\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2)$ and therefore to $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$ by the previous paragraph, and tends to z and $\varepsilon \to 0$. □

In order to prove the theorem, we need to build inside any non-classical cone something which looks like entanglement. Here is the key gimmick. To get some intuition, we understand the simplest non-classical cone.

Consider a 3-dimensional vector space V spanned by vectors $x_1, x_0, x_\oplus, x_\ominus$ such that $x_0 + x_1 = x_\oplus + x_\ominus$. Let $\mathsf{C}_\square$ be the cone generated by these vectors. The cone $\mathsf{C}_\square$ is isomorphic to its dual cone, but in a non-canonical way. An isomorphism is given by the map $\Theta : V^* \to V$

$$\Theta(f) = f(x_0)x_1 + f(x_\oplus)(x_0 - x_\oplus) + f(x_1)x_\oplus.$$

Indeed, the cone $\mathsf{C}_\square^*$ has 4 extremal elements, corresponding to generators $f_{0\oplus}$, $f_{1\oplus}$, $f_{0\ominus}$, $f_{1\ominus}$ given by $f_{0\oplus}(x_0) = f_{0\oplus}(x_\oplus) = 0$ and $f_{0\oplus}(x_1) = f_{0\oplus}(x_\ominus) = 1$ and so on. One check that the formula is correct since

$$\Theta(f_{0\oplus}) = x_\oplus,\ \Theta(f_{0\ominus}) = x_0,\ \Theta(f_{1\oplus}) = x_1,\ \Theta(f_{1\ominus}) = x_1 + x_0 - x_\oplus = x_\ominus$$

Proposition 5. *The cone $\mathsf{C}_\square \otimes_{\max} \mathsf{C}_\square$ has 24 extreme rays*

- *16 extreme rays of rank 1, of the form $x_i \otimes x_j$,*
- *8 extreme rays of rank 3, of the form*

$$x_1 \otimes x_a + (x_0 - x_\oplus) \otimes x_b + x_\oplus \otimes x_c$$

where x_a, x_b, x_c are consecutive vertices of the quadrangle $\{x_0, x_\oplus, x_1, x_\ominus\}$.

Exercise 5. If x_1 is extremal in C_1 and x_2 is extremal in C_2, then $x_1 \otimes x_2$ is extremal in $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$ and in $\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$.

We first consider the cone $\mathsf{P}(\mathsf{C}_\square, \mathsf{C}_\square)$ of positive maps from $\mathsf{C}_\square$ to $\mathsf{C}_\square$. This cone is canonically isomorphic to $\mathsf{C}_\square^* \otimes_{\max} \mathsf{C}_\square$, and an isomorphism from $\mathsf{P}(\mathsf{C}_\square, \mathsf{C}_\square)$ to $\mathsf{C}_\square \otimes_{\max} \mathsf{C}_\square$ is given by

$$\Phi \mapsto \Phi(x_0) \otimes x_1 + \Phi(x_\oplus) \otimes (x_0 - x_\oplus) + \Phi(x_1) \otimes x_\oplus$$

The cone $\mathsf{P}(\mathsf{C}_\square, \mathsf{C}_\square)$ is isomorphic to $\mathsf{C}_\square^4 \cap H$, where $H \subset V^4$ is the subspace given by quadruples (a, b, c, d) such that $a + c = b + d$. Let Φ be an extreme ray generator. We show that Φ has either rank 1 or corresponds to a symmetry of the square, of the form

$$\Phi(x_0) = \lambda y_a,\quad \Phi(x_\oplus) = \lambda y_b,\quad \Phi(x_1) = \lambda y_c$$

(i) $\underline{\operatorname{rank}\Phi = 3}$. We claim that Φ maps extreme rays to extreme rays. Once the claim is proved, we obtain that $\Phi(x_i) = \lambda_i x_{\sigma(i)}$ for a permutation $\sigma \in \mathfrak{S}_4$ and $\lambda_i > 0$. The linearity of Φ implies that $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, and it is clear that Φ must be of the form given.

To prove the claim, assume by contradiction that $\Phi(x_i)$ is not on an extreme ray for some i; without loss of generality assume $i = \ominus$. There exists z not collinear with $\Phi(x_\ominus)$ such that $\Phi(x_\ominus) + \delta z \in \mathsf{C}$ for $|\delta|$ small enough. Because Φ has full rank, we may write $z = \alpha_0 \Phi(x_0) + \alpha_1 \Phi(r_1) + \alpha_\oplus \Phi(r_\oplus)$, and the vector α is not collinear to $(1, -1, 1)$. For δ small enough (positive or negative), the map given by

$$x_1 \mapsto (1+\delta\alpha_1)\Phi(x_1),\ \ x_2 \mapsto (1-\delta\alpha_2)\Phi(x_2),\ \ x_3 \mapsto (1+\delta\alpha_3)\Phi(x_3),\ \ x_4 \mapsto \Phi(x_4)+\delta z$$

is positive; showing that Φ is not extremal — a contradiction.

(ii) $\underline{\operatorname{rank}\Phi \leqslant 2}$. By Cariello's theorem, such a Φ must be of rank 1 and of the form given by the Proposition.

3. Lecture 3

3.1. **Square states.** Given a proper cone $\mathsf{C} \subset V$, a family of *square states* for C is a quadruple $(x_0, x_1, x_\oplus, x_\ominus)$ of nonzero elements in C such that

(1) $x_0 + x_1 = x_\oplus + x_\ominus$,
(2) There exists a quadruple $(f_0, f_1, f_\oplus, f_\ominus) \in (\mathsf{C}^*)^4$ such that
 (a) $f_0 + f_1 = f_\oplus + f_\ominus$,
 (b) $f_0(x_0) = f_1(x_1) = f_\oplus(x_\oplus) = f_\ominus(x_\ominus) = 0$,
 (c) $f_i + f_j \in \mathrm{int}(\mathsf{C}^*)$ for any $i \neq j$.

In this definition, we may weaken (1) and (2a) to

$$[x_0, x_1] \cap [x_\oplus, x_\ominus] \neq \varnothing, \quad [f_0, f_1] \cap [f_\oplus, f_\ominus] \neq \varnothing,$$

since we may then replace (x_i) and (f_i) by suitable multiples to obtain (1) and (2a).

Example 3. If $\mathsf{C} = \mathsf{PSD}(\mathbf{C}^2)$, a family of square states is given by

$$x_0 = |0\rangle\langle 0|, \ x_1 = |1\rangle\langle 1|, \ x_\oplus = |+\rangle\langle +|, \ x_\ominus = |-\rangle\langle -|$$

with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

We are going to prove separately the following two results

Theorem 2 (Theorem A)**.** *A proper cone C is nonclassical iff there exists a family of square states for C.*

One direction is easy: a classical cone does not have square-like states. Indeed assume that $x_0 + x_1 = x_\oplus + x_\ominus$ with (x_i) in a classical cone C. By the decomposition property (which can be proved coordinatewise, reducing in a 1-dimensional problem), there exist $x_{0\oplus}, x_{0\ominus}, x_{1\oplus}, x_{1\ominus}$ in C such that

$$x_0 = x_{0\oplus} + x_{0\ominus}, \ x_1 = x_{1\oplus} + x_{1\ominus}, \ x_\oplus = x_{0\oplus} + x_{1\oplus}, \ x_\ominus = x_{0\ominus} + x_{1\ominus}$$

We have $f_1(x_{0\oplus}) \leqslant f_1(x_0) = 0$ and so on, so $(f_1 + f_\ominus)(x_{0\oplus}) = 0$ and therefore $x_{0\oplus} = 0$. By similar arguments, we come quickly to the fact that $x_0 = x_1 = x_\oplus = x_\ominus = 0$.

Theorem 3 (Theorem B)**.** *Let C_1 and C_2 be two classical cones, and (x_i) be square states for C_1 and (y_i) be square states for C_2. The element*

$$\omega = x_0 \otimes y_\oplus - x_\oplus \otimes y_\oplus + x_\oplus \otimes y_0 + x_1 \otimes y_1$$

belongs to $\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$ but not to $\mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$, and therefore the pair $(\mathsf{C}_1, \mathsf{C}_2)$ is entangleable.

4. Proof of Theorem A

4.1. **Affine diameters.** A segment $[a, b]$ of a convex body $K \subset \mathbf{R}^n$ is an affine diameter if there exists a nonconstant linear form g which is maximal on K at a and minimal on K at b.

Lemma 4 (Hammer, 1963)**.** *If $n \geqslant 1$, $K \subset \mathbf{R}^n$ is a convex body and any $z \in K$, there is an affine diameter $[a, b]$ for K such that $z \in [a, b]$.*

Proof. Take $r \geqslant 0$ maximal such that $-rK \subset K$. By maximality, there is a point $a \in \partial(-rK) \cap \partial K$, and there is a nonzero linear form f which is maximal at a on both $-rK$ and K. If $b = -ra$, then $[a, b]$ is an affine diameter containing 0. □

4.2. **Exposed vs extreme points.** A face F of a convex body K is a convex subset $F \subset K$ such that if $x \in F$ can be written as $x = \lambda x' + (1-\lambda)x''$ then $x', x'' \in F$. An exposed face is a face of the form $K \cap H$ where H is a tangent hyperplane. Not every face is exposed (stadium example), but almost. Say that $x \in K$ is a d-extreme point (resp. d-exposed point) if it belongs to a d-dimensional face (resp. exposed face). Then

Theorem 5 (Straszewicz ($d = 0$), Asplund)**.** *Any d-extreme point is the limit of a sequence of d-exposed points.*

We now complete the proof and introduce the parameter $\delta(K)$ as the minimum dimension d such the convex hull of an extreme point and a d-extreme point intersects $\mathrm{int}(K)$. By the Straszewicz–Asplund theorem, we may replace extreme by exposed in this definition. The key lemma (not proved here) is

Lemma 6. *If K is a convex body in $\mathbf{R}^n$ which is not a simplex, then $\delta(K) < n-1$.*

4.3. **Proof of Theorem A.** Let C be a non-classical cone with $K = \mathsf{C} \cap H$ as a base, with $H = \{h = 1\}$ an affine hyperplane. Set $d = \delta(K)$. Up to replacing K by a projective image, we may assume that there is a linear form f such that

$$f(x_0) = \min_K f < \max_K f$$

with $x_0 \in K$ an exposed point and $F = K \cap f^{-1}(\alpha_1)$ an exposed face of dimension d. We denote by $W \subset H$ the affine subspace generated by x_0 and F. We have $\dim W = d + 1 < \dim(H)$. One can check (using maximality in the definition of δ) $W \cap K = \mathrm{conv}(x_0, F)$.

Let p be an affine map on H such that $p^{-1}(z) = W$, so the rank of p is $\geqslant 1$. Observe that $z \in \mathrm{int}(p(K))$. By Hammer's lemma, there is a nonconstant affine map γ such that minimal on $p(K)$ at $y_\ominus$ and maximal at $y_\oplus$, such that $z = \lambda y_\ominus + (1-\lambda)y_\oplus$. Write $y_\ominus = p(x_\ominus)$ and $y_\oplus = p(x_\oplus)$. The point $x = \lambda x_\ominus + (1-\lambda)x_\oplus$ satisfies $p(x) = z$ hence belongs to $K \cap W$ and can be written as $\mu x_0 + (1-\mu)x_1$ for some $x_1 \in F$. This gives square states for C.

If we denote $g = \gamma \circ p$, then

$$g(x_\ominus) = \min_K g < g(x_0) = g(x_1) < \max_K g = g(x_\oplus).$$

This gives square states for C: consider the functionals

$$f_0 = f - f(x_0)h,\ f_1 = f(x_1)h - f,\ f_\ominus = g - g(x_\ominus),\ f_\oplus = g(x_\oplus) - g$$

4.4. **Proof of Theorem B.** To show that ω does not belong to $\mathsf{C}_1 \otimes_{\max} \mathsf{C}_2$, we construct a Bell inequality.

Let's start with the most famous Bell inequality.

Lemma 7. *If $|s_1| \leqslant A_1$, $|t_1| \leqslant A_1$, $|s_2| \leqslant A_2$, $|t_2| \leqslant A_2$ then*

$$|s_1 s_2 + s_1 t_2 + t_1 s_2 - t_2 t_2| \leqslant 2A_1 A_2.$$

If moreover $(|s_1|, |t_1|) \neq (A_1, A_1)$ and $(|s_2|, |t_2|) \neq (A_2, A_2)$ then

$$|s_1 s_2 + s_1 t_2 + t_1 s_2 - t_2 t_2| < 2A_1 A_2.$$

Given α_1, β_1 in V_1^* and α_2, β_2 in V_2^*, we may define

$$\mathrm{CHSH}(\alpha_1, \beta_1, \alpha_2, \beta_2) = \alpha_1 \otimes \alpha_2 + \alpha_1 \otimes \beta_2 + \beta_1 \otimes \alpha_2 - \beta_1 \otimes \beta_2 \in V_1^* \otimes V_2^*.$$

Let $(f_i) \in \mathsf{C}_1^*$ and $(g_i) \in \mathsf{C}_2^*$ as in the definition of square-like states. We consider the linear form

$$\lambda = 2(f_0 + f_1) \otimes (g_0 + g_1) - \mathrm{CHSH}(f_0 - f_1, f_\oplus - f_\ominus, g_0 - g_1, g_\oplus - g_\ominus)$$

For every $x_1 \in \mathsf{C}_1 \backslash \{0\}$ and $x_2 \in \mathsf{C}_2 \backslash \{0\}$, we have

$$|(f_0 - f_1)(x_1)| \leqslant A_1, \quad |(f_\oplus - f_\ominus)(x_1)| \leqslant A_1 \tag{1}$$

$$|(g_0 - g_1)(x_2)| \leqslant A_2, \quad |(g_\oplus - g_\ominus)(x_2)| \leqslant A_2 \tag{2}$$

for

$$A_1 = (f_0 + f_1)(x_1) = (f_\oplus + f_\ominus)(x_1),$$
$$A_2 = (g_0 + g_1)(x_2) = (g_\oplus + g_\ominus)(x_2).$$

Moreover, since at most one of the numbers $(f_i(x_1))$ is zero, at least one inequality in (1) is strict. The same applies to (2). It follows from Lemma 7 that $\lambda(x_1 \otimes x_2) > 0$. It follows that $\lambda(z) > 0$ for any nonzero $z \in \mathsf{C}_1 \otimes_{\min} \mathsf{C}_2$.

On the other hand, a very long but not difficult computation shows that

$$\lambda(\omega) = 4(f_0(x_\oplus) - f_\oplus(x_0))(g_0(y_\oplus) - g_\oplus(y_0)).$$

We may assume, up to replacing $(0, 1, \oplus, \ominus)$ by $(\oplus, \ominus, 0, 1)$, that $\lambda(\omega) \leqslant 0$.