# Averaging over local unitary groups in Random Tensor Networks

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*Random tensors and related topics*

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# Outline

- Generalities on entanglement and local unitary invariants

- Review of standard results on RTNs

- RTNs with reduced randomness: local Haar-averaging

[Work in progress with Luca Lionni + ...]

# Generalities on entanglement and local unitary invariants

# Separable and entangled states

$$\boxed{\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_q}$$

**Pure states.** We say that $|\psi\rangle \in \mathcal{H}$ is

- *separable* if $\exists \{|v_i\rangle\}$ such that

$$|\psi\rangle = |v_1\rangle \otimes |v_2\rangle \otimes \cdots \otimes |v_q\rangle$$

- *entangled* otherwise.

**Mixed states.** $\rho$ is

- *separable* if $\exists \{\rho_i^{(k)}\}$ and $\{\alpha_i | \alpha_i \geq 0, \sum_i \alpha_i = 1\}$ such that

$$\rho = \sum_k \alpha_k \rho_1^{(k)} \otimes \rho_2^{(k)} \otimes \cdots \otimes \rho_q^{(k)}$$

- *entangled* otherwise.

# Local unitary symmetry

$$\boxed{\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_q}$$

**Local Unitary (LU) transformation.** Change of frame in each local subsystem $\mathcal{H}_i$, represented by

$$U_1 \otimes U_2 \otimes \cdots \otimes U_q \qquad \text{with} \qquad U_i \in \mathrm{U}(D_i) \quad \forall i$$

***Entanglement structure.*** Orbit under action of $\mathrm{U}(D_1) \otimes \mathrm{U}(D_2) \otimes \cdots \otimes \mathrm{U}(D_q)$ i.e. equivalence class of states:

- *pure states*:

$$\boxed{|\psi\rangle \sim_{\mathrm{LU}} |\psi'\rangle \qquad \Leftrightarrow \qquad |\psi'\rangle = (U_1 \otimes U_2 \otimes \cdots \otimes U_q)\,|\psi\rangle}$$

*Example.* {separable states} constitute one such equivalent class.

- *mixed states*:

$$\rho \sim_{\mathrm{LU}} \rho' \quad \Leftrightarrow \quad \rho' = (U_1 \otimes U_2 \otimes \cdots \otimes U_q)\,\rho\,(U_1 \otimes U_2 \otimes \cdots \otimes U_q)^{-1}$$

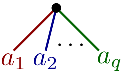
# LU INVARIANTS FROM COLORED GRAPHS

From now on, restrict to pure state:

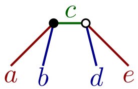
$$|\psi\rangle = \sum_{a_1, a_2 \cdots a_q} \underbrace{T_{a_1 a_2 \cdots a_q}}_{\text{tensor}} |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_q\rangle$$

Represent tensors and tensor contractions by *q-colored graphs*:

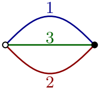
$$T_{a_1 a_2 \cdots a_q} =$$



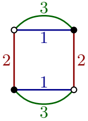

$$\sum_c T_{abc} \overline{T_{cde}} =$$




Any *closed q-colored graph* $\mathcal{B}$ represents a *LU-invariant polynomial*, which we denote $\mathrm{Tr}_{\mathcal{B}}(\bar{T}, T)$.




$$\leftrightarrow \overline{T_{abc}} T_{abc}$$



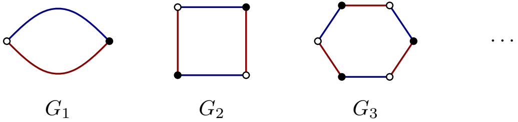

$$\leftrightarrow \overline{T_{abc}} T_{adc} \overline{T_{edf}} T_{ebf}$$

**Claim.** $\{\mathrm{Tr}_{\mathcal{B}}(\bar{T}, T) \,|\, \mathcal{B} \text{ connected}\}$ generates the ring of polynomial LU-invariants. Any two $\mathcal{B}_1 \neq \mathcal{B}_2$ produce independent invariants in the limit of large dimension ($D_1, \ldots, D_q \to +\infty$). [R. Gurau's talk]

# Entanglement spectrum of a bipartite system

$$|\psi\rangle = \sum_{a_1,a_2} \underbrace{T_{a_1 a_2}}_{\text{rectangular matrix}} |a_1\rangle \otimes |a_2\rangle$$

$\exists$ a single connected invariant of order $2n$, represented by cyclic graph $G_n$:




$$\boxed{\mathrm{Tr}_{G_n}(\bar{T}, T) = \mathrm{tr}((TT^\dagger)^n) = \mathrm{tr}(\rho_1^n) = \sum_i p_i^n}$$

Entanglement structure of $T$ characterized by its *entanglement spectrum*:

$$\{\text{LU invariants of } |\psi\rangle\} \quad \Leftrightarrow \quad \{\mathrm{tr}\left((TT^\dagger)^n\right)\}_{1\leq n\leq \min(D_1,D_2)}$$

$$\Leftrightarrow \quad \mathrm{Spec}(\mathrm{TT}^\dagger) = \{\mathrm{p_i}\}_{1\leq \mathrm{i}\leq \min(\mathrm{D_1},\mathrm{D_2})}$$

$$\Leftrightarrow \quad \{\text{singular values of } T\} = \{\sqrt{p_i}\}_{1\leq i\leq \min(D_1,D_2)}$$

*Remark.* Singular value decomposition $\Rightarrow$ "Schmidt decomposition" :

$$|\psi\rangle = \sum_{i=1}^{\min(D_1,D_2)} \sqrt{p_i}|e_i\rangle \otimes |f_i\rangle$$

# Entanglement entropies

Functions of the entanglement spectrum which have special properties (e.g. monotonicity under quantum operations) [P. Vrana's talk]

**Entanglement entropy.** *Von Neumann entropy* of $\rho_1$ (or $\rho_2$)

$$S(\rho_1) = -\mathrm{tr}(\rho_1 \ln(\rho_1)) = -\sum_{i=1}^{\min(D_1,D_2)} p_i \ln(p_i)$$

**Rényi-$n$ quantum entropy.** *Rényi-n entropy* $\rho_1$ (or $\rho_2$)

$$S_n(\rho_1) = \frac{1}{1-n} \ln\left(\mathrm{tr}({\rho_1}^n)\right) = \frac{1}{1-n} \ln\left(\sum_{i=1}^{\min(D_1,D_2)} p_i^n\right)$$

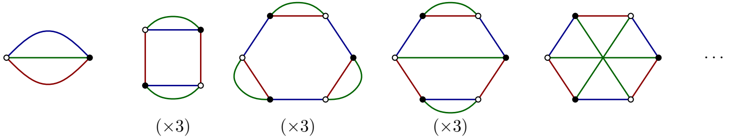
*Example.* Maximally entangled state / "Bell state"

$$|\psi\rangle = \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle \otimes |i\rangle \quad \Rightarrow \quad S_n(|\psi\rangle) = \ln D$$

$\rightarrow$ *flat* entanglement spectrum.

# Multipartite entanglement and tensor invariants

$$|\psi\rangle = \sum_{a_1, a_2 \cdots a_q} \underbrace{T_{a_1 a_2 \cdots a_q}}_{\text{tensor}} |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_q\rangle \,, \qquad \boxed{q \geq 3}$$




$$\#\{\text{connected invariants of order 2n}\} = 1\,,\ 3\,,\ 7\,,\ 26\,,\ 97\,,\ 624\,,\ 4163\ldots$$

Super-exponential growth of the number of independent invariants!

[Ben Geloun, Ramgoolam '13]

Question

Which of those many invariants are most relevant to characterize the multipartite entanglement structure of a many-body system?

# Tensor invariants as permutations

Invariants can be conveniently parametrized by $q$-uplet of permutations $(\tau_1, \tau_2, \ldots, \tau_q) \in S_n^{\times q}$ [Ben Geloun, Ramgoolam '13]

$$\{q - \text{colored graph of order } 2n\} \underset{1-1}{\longleftrightarrow} S_n \backslash S_n^{\times q} / S_n$$


$\vec{\tau} = (\mathrm{id}, \mathrm{id}, \mathrm{id})$ $\qquad$ $\vec{\tau} = ((12), \mathrm{id}, \mathrm{id})$ $\qquad$ $(n = 2\,, q = 3)$

With this convention in mind, we will write:

$$\mathrm{Tr}_{\mathcal{B}}(\bar{T}, T) = \mathrm{Tr}_{\vec{\tau}}(\bar{T}, T)$$

*Remark.* $(q = 2)$ $\mathrm{tr}({\rho_1}^n) = \mathrm{Tr}_{G_n}(\bar{T}, T) = \mathrm{Tr}_{(\mathrm{id}, \sigma)}(\bar{T}, T)$, $\sigma := (12 \cdots n)$.

"Replica trick"

# Review of some standard results on RTNs

1. Page curve
2. Area laws
3. RTNs with holographic area laws

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# Random bipartite state: definition

$$|\psi\rangle = \sum_{i,j} M_{ij} \, |i\rangle \otimes |j\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

Two equivalent ways of defining a *uniform random state*:

1. $|\psi\rangle = U|0\rangle$ with $U$ random Haar-distributed on $\mathrm{U}(D_1 D_2)$
2. $M_{ij}$ Gaussian random matrix with covariance

$$\langle M_{ij} \bar{M}_{kl} \rangle = \frac{1}{D_1 D_2} \delta_{ik} \delta_{jl}$$

   so that $\langle \mathrm{tr}(M M^\dagger) \rangle = 1$.

For simplicity, let us adopt the second option, assuming $\boxed{D_1 D_2 \gg 1}$ (no further normalization required in this regime).

Goal: compute $\langle \mathrm{tr}({\rho_1}^n) \rangle = \langle \mathrm{Tr}_{(\mathrm{id},\sigma)}(\bar{T}, T) \rangle$, and deduce typical entanglement spectrum / entanglement entropies.

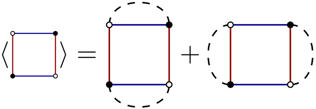
# Random bipartite state: $D_1 \ll D_2$ regime

Recall Wick's formula:

$$\langle M_{i_1 j_1} \bar{M}_{k_1 l_1} M_{i_2 j_2} \bar{M}_{k_2 l_2} \cdots M_{i_n j_n} \bar{M}_{k_n l_n} \rangle = \frac{1}{(D_1 D_2)^n} \sum_{\tau \in S_n} \prod_{p=1}^{n} \delta_{i_p k_{\tau(p)}} \delta_{j_p l_{\tau(p)}}$$

To simplify our life even more, let us first assume $D_1 \ll D_2$.

$$\langle \mathrm{tr}({\rho_1}^2) \rangle = \langle \mathrm{Tr}_{\square}(\bar{M}, M) \rangle =$$



$$= \frac{1}{(D_1 D_2)^2} \left( D_1 {D_2}^2 + {D_1}^2 D_2 \right) \approx \frac{1}{D_1}$$

More generally, there is a single dominant Wick contraction for $G_n$:

$$\langle \mathrm{tr}({\rho_1}^n) \rangle \approx \frac{1}{(D_1 D_2)^n} D_1 {D_2}^n = {D_1}^{1-n}$$

# RANDOM BIPARTITE STATE: $D_1 \ll D_2$ REGIME

Here, one can prove that

$$\ln\langle \mathrm{tr}({\rho_1}^n)\rangle \approx \langle \ln \mathrm{tr}({\rho_1}^n)\rangle$$

so that

$$\langle S_n(\rho_1)\rangle \approx \frac{1}{1-n}\ln\left({D_1}^{1-n}\right) \approx \ln(D_1)$$

Taking the limit $n \to 1$ (can be justified...), we deduce

$$\boxed{\langle S(\rho_1)\rangle \approx \ln(D_1)}$$

**Volume law**

Is this behaviour more generally valid for $D_1 \leq D_2$?

# Random bipartite state: $D_1 \leq D_2$ regime

$$\langle \mathrm{tr}({\rho_1}^n) \rangle = \langle \mathrm{Tr}_{(\mathrm{id},\sigma)}(\bar{T}, T) \rangle \approx \frac{1}{(D_1 D_2)^n} \sum_{\tau \in S_n} {D_1}^{C(\tau)} {D_2}^{C(\tau^{-1} \circ \sigma)}$$

where

- $C(\tau) = \#$ cycles in the cycle-decomposition of $\tau$;
- $\sigma = (12 \cdots n)$.

In the regime $D_1 D_2 \to +\infty$ with $\frac{D_1}{D_2}$ fixed, the leading contributions maximize $C(\tau) + C(\tau^{-1} \circ \sigma)$.

Claim. This happens when

$$\boxed{C(\tau) + C(\tau^{-1} \circ \sigma) = n + 1}$$

*Proof.* $C(\tau) + C(\tau^{-1} \circ \sigma)$ counts the number of faces of a *combinatorial map* (*discrete surface*) with $n$ edges and 1 vertex. We must therefore have

$$C(\tau) + C(\tau^{-1} \circ \sigma) = n + 1 - 2g$$

where $g \geq 0$ is the genus of the surfaces. □

# Random bipartite state: $D_1 \leq D_2$ regime

Def. A *non-crossing permutation* is a $\tau \in S_n$ obeying:

$$C(\tau) + C(\tau^{-1} \circ \sigma) = n + 1$$

$$\Leftrightarrow \quad \boxed{d(\mathrm{id}, \tau) + d(\tau, \sigma) = d(\mathrm{id}, \sigma)}$$

where $d(\tau_1, \tau_2) := n - C(\tau_1 \circ \tau_2^{-1})$ is the *Cayley distance* on $S_n$. One also says that $\tau$ is *on a geodesic* between id and $\sigma$.

With these definitions, one finds

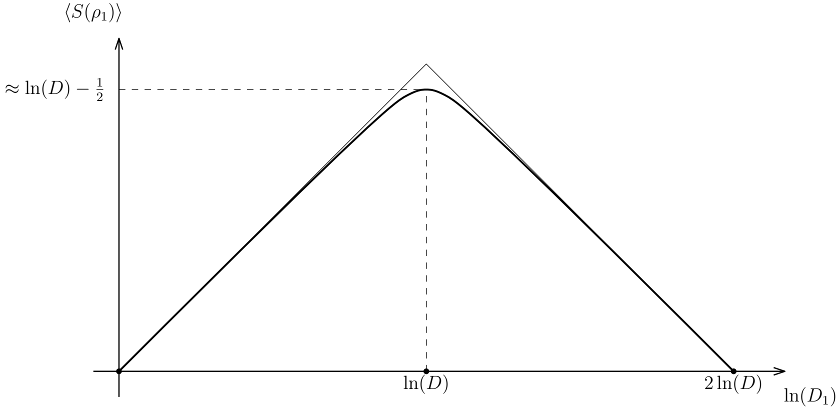
$$\langle \mathrm{tr}(\rho_1{}^n) \rangle \approx D_1{}^{1-n} \sum_{\tau \in S_n \text{ n.c.}} \left( \frac{D_1}{D_2} \right)^{C(\tau)-1} = D_1{}^{1-n} \, {}_2F_1(1-n, -n, 2; \frac{D_1}{D_2})$$

After analytic continuation of $n$, we can compute $\lim_{n \to 1} \frac{1}{1-n} \ln \langle \mathrm{tr}(\rho_1{}^n) \rangle$, yielding

$$\boxed{\langle S(\rho_1) \rangle \approx \underbrace{\ln(D_1)}_{\text{volume law}} - \underbrace{\frac{D_1}{2D_2}}_{\text{finite correc.}}}$$

# Random bipartite state: Page curve

Set $D^2 := D_1 D_2$.




[Page '93; Foong, Kano '94; Sánchez-Ruiz '95; Sen '96]

# Review of some standard results on RTNs

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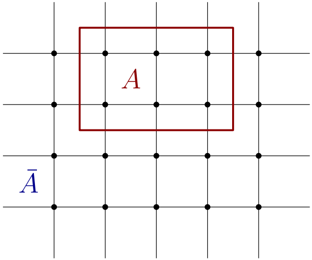
# Condensed matter

[Talks by N. Schuch and M. C. Bañuls]



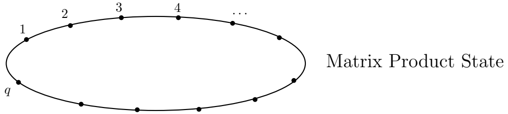

For a random $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$:

$$\langle S(\rho_A)\rangle \sim \ln(d_{\text{loc}}^{V_A}) \sim V_A$$

However, the ground state of a gapped local Hamiltonian typically obeys an area law

$$\boxed{S(\rho_A) \leq K|\partial A|}$$

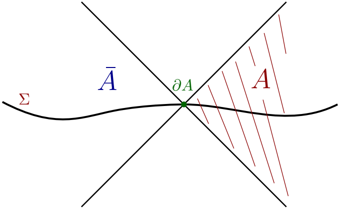
$\longrightarrow$ Tensor Networks (TN): variational Ansätze with a polynomial number of parameters in system size, which obey area laws by design.




$$|\psi\rangle = \sum_{a_1, a_2 \cdots a_q} \text{tr}(M^{a_1} M^{a_2} \cdots M^{a_q}) |a_1\rangle \otimes \cdots \otimes |a_q\rangle$$

# Relativistic QFT

Unruh effect / Bisognano-Wichman theorem (and, similarly, BH entropy)




$$"\rho_A = \frac{e^{-2\pi H_A}}{Z}"$$

with $H_A$ = boost Hamiltonian.

*Universal divergence*

$$S(\rho_A) \propto K|\partial A|$$

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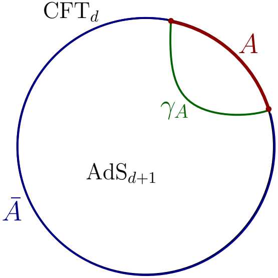
Suggested (heuristic) entropic derivation of general relativity [Jacobson '95, '15]

$$\begin{cases} \delta S(\rho_A) = \underbrace{\eta}_{\text{universal}} \delta|\partial A| \\ \text{"entanglement equilibrium hypothesis"} \end{cases} \iff G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$$

# Holography

Holographic area law (AdS/CFT):




Ryu-Takayanagi formula

$$S(\rho_A) = \frac{|\partial A|}{4G}$$

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Holographic version of Jacobson's argument:

Ryu-Takayanagi for any ball $A \Rightarrow$ Einstein's equations up to 2nd order

[Faulkner et al. 2017]

# Review of some standard results on RTNs

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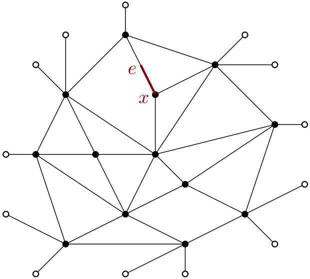
# Definition

[M. Walter's talk]

Graph $G = (V, E)$ with $V = V_{\text{bulk}} \sqcup V_\partial = \{\bullet\} \sqcup \{\circ\}$




Associate Hilbert space $\mathcal{H}_{e,x}$ to each half-edge $(e, x)$, with

$$\boxed{\dim \mathcal{H}_{e,x} = D} \quad \text{(bond dimension)}$$

Edge data. Maximally mixed bipartite state along each edge

$$|\phi\rangle = \bigotimes_{(x,y)\in E} \frac{1}{\sqrt{D}} \sum_{i=1}^{D} |i\rangle_x \otimes |i\rangle_y$$

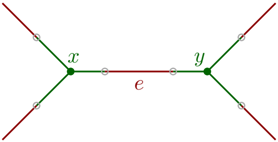
Vertex data.

$$|\eta\rangle = \bigotimes_{x\in V_{\text{bulk}}} |\eta\rangle_x\,, \qquad |\eta\rangle_x \in \bigotimes_{e \text{ incident to } x} \mathcal{H}_{e,x}$$

# Definition

Tensor Network state:

$$\boxed{|\psi\rangle = (\mathrm{Id}_{V_\partial} \otimes \langle\eta|)\,|\phi\rangle} \in \bigotimes_{v \in V_\partial} \mathcal{H}_{e_v,v}$$

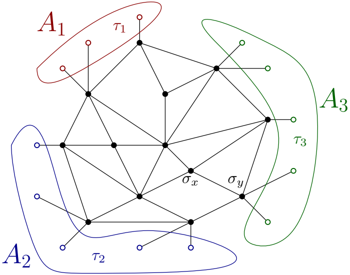



***Random* Tensor Network state.** Take $\{|\eta\rangle_x\}$ to be independent, Haar-distributed (or Gaussian) random vectors. We then have the Wick formula

$$\boxed{\mathbb{E}\left(|\eta\rangle_{xx}\langle\eta|^{\otimes n}\right) \propto \sum_{\sigma \in S_n} R_x(\sigma)}$$

where $R_x(\sigma)$ is a permutation operator acting on $\mathcal{H}_x^{\otimes n}$

# Expectation values of LU invariants




Define

$$\mathcal{Z}_{(\tau_1,\tau_2,\ldots,\tau_q)} := \langle \mathrm{Tr}_{(\tau_1,\tau_2,\ldots,\tau_q)}(\bar{T}, T) \rangle$$

Partition function of generalized spin model

$$\mathcal{Z}_{(\tau_1,\tau_2,\ldots,\tau_q)} = \sum_{\substack{\sigma:V\to S_n \\ \forall v\in A_k\,,\sigma(v)=\tau_k}} \exp\left(-\beta\mathcal{E}(\sigma)\right)$$

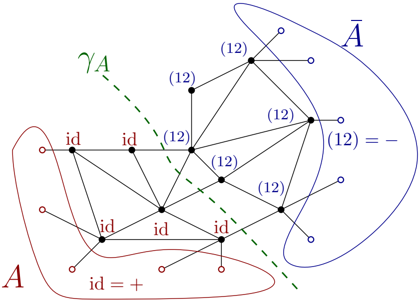
$$\mathcal{E}(\sigma) = \sum_{(x,y)\in E} d(\sigma(x),\sigma(y)) \qquad \beta = \ln(D)$$

# Entropy ∼ ground state energy

The $D \to +\infty$ limit is a *zero-temperature limit*. At leading-order, we thus have:

$$\boxed{\mathcal{Z}_{(\tau_1,\tau_2,\ldots,\tau_q)} = \mathcal{N}_{g.s.} D^{-\mathcal{E}_{\min}} \left(1 + \mathcal{O}(1/D)\right)}$$

*Example.* $\mathcal{Z}_{(\mathrm{id},(12))} = \langle \mathrm{tr}(\rho_A^2) \rangle$ is a Ising partition function.




Let $\gamma_A$ be a *min-cut*, that is: a minimal collection of edges that disconnect $A$ from $\bar{A}$ when cut.

$$\mathcal{E}_{\min} = |\gamma_A|$$

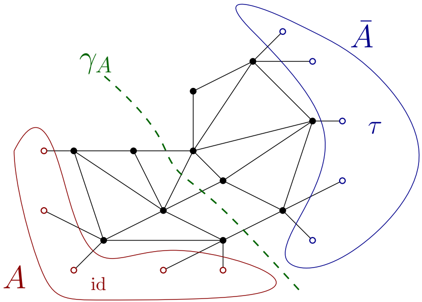
$$\langle \mathrm{tr}(\rho_A^2) \rangle \approx \mathcal{N}_{g.s.} D^{-|\gamma_A|}$$

*Holographic area law* for Rényi-2 entropy:

$$\boxed{\langle S_2(\rho_A) \rangle \approx |\gamma_A| \ln(D)}$$

# Bipartite setting: general results

**Min-cut / Max-flow theorem.** (simplified version)
There exists $|\gamma_A|$ edge-disjoint paths connecting $A$ to $\bar{A}$.




*Proof.* Explicit algorithm: Ford-Fulkerson (50's).

**Theorem.** We have $\mathcal{Z}_{(\mathrm{id},\tau)} \approx \mathcal{N}_{g.s.} D^{-d(\mathrm{id},\tau)|\gamma_A|}$. In particular, taking $\tau = (12 \cdots n)$ leads to

$$\boxed{\langle S_n(\rho_A) \rangle \approx |\gamma_A| \ln(D)}$$

[Collins, Nechita, Zyczkowski '13 ; Hayden et al '16; ...]

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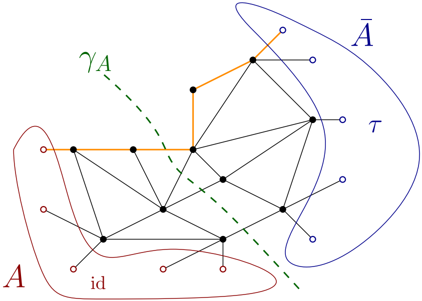


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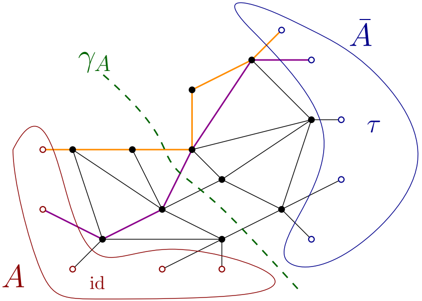


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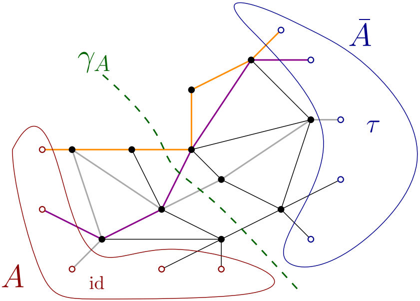


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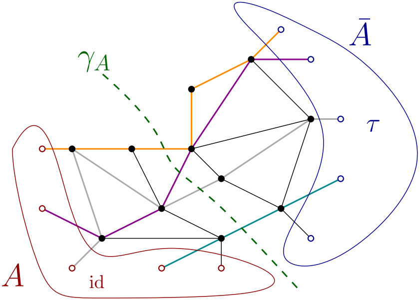
**Min-cut / Max-flow theorem.** (simplified version)
There exists $|\gamma_A|$ edge-disjoint paths connecting $A$ to $\bar{A}$.




*Proof.* Explicit algorithm: Ford-Fulkerson (50's).

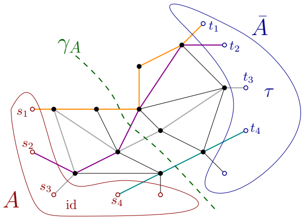
**Theorem.** We have $\mathcal{Z}_{(\mathrm{id},\tau)} \approx \mathcal{N}_{g.s.} D^{-d(\mathrm{id},\tau)|\gamma_A|}$. In particular, taking $\tau = (12\cdots n)$ leads to

$$\boxed{\langle S_n(\rho_A)\rangle \approx |\gamma_A| \ln(D)}$$

[Collins, Nechita, Zyczkowski '13 ; Hayden et al '16; ...]

# Bipartite setting: general results

*Proof.* Repeated application of the triangular inequality along each flow.



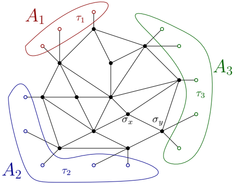

Let $\mathcal{P} = \mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_{|\gamma_A|}$ the edges of a max-flow.

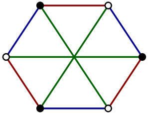
$$\mathcal{E}(\sigma) = \sum_{(x,y)\in E} d(\sigma(x), \sigma(y)) \geq \sum_{i=1}^{|\gamma_A|} \sum_{(x,y)\in\mathcal{P}_i} d(\sigma(x), \sigma(y))$$

$$\geq \sum_{i=1}^{|\gamma_A|} d(\sigma(s_i), \sigma(t_i)) = |\gamma_A| d(\mathrm{id}, \tau)$$

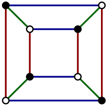
And those inequalities are saturated. □

# Multipartite setting: two examples



(id, (123), (132))



((12)(34), (13)(24), (14)(23))

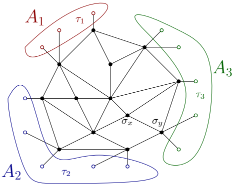
**Definition.** $(\tau_1, \tau_2, \tau_3)$ is *compatible* if $\exists\ \sigma$ such that:

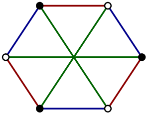
- $d(\tau_1, \sigma) + d(\sigma, \tau_2) = d(\tau_1, \tau_2)$;
- $d(\tau_1, \sigma) + d(\sigma, \tau_3) = d(\tau_1, \tau_3)$;
- $d(\tau_2, \sigma) + d(\sigma, \tau_3) = d(\tau_2, \tau_3)$.

**Claim.**

- (id, (123), (132)) is *compatible*;
- ((12)(34), (13)(24), (14)(23)) is *not compatible*.

# Multipartite setting: two examples



$(\text{id}, (123), (132))$

**Theorem.** For $\vec{\tau} = (\text{id}, (123), (132))$

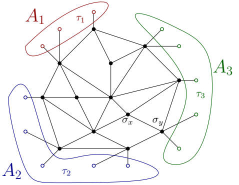
$$\boxed{\mathcal{E}_{\min} = |\gamma_{A_1}| + |\gamma_{A_2}| + |\gamma_{A_3}|}$$

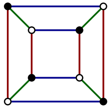
*Proof.* Generalization of previous argument relying on *multi-cut / max-flow theorem...* □

[Cui, Hayden, et al '18 ; Dong, Qi, Walter '21; Kudler-Flam, Ryu, Narovlansky '21]

→ Entanglement measure which, at leading order, does not capture information that was not already contained in bipartite measures.

# Multipartite setting: two examples



$((12)(34), (13)(24), (14)(23))$

**Theorem.** For $\vec{\tau} = ((12)(34), (13)(24), (14)(23))$

$$\boxed{\mathcal{E}_{\min} = 4 \min_{\text{tripartition } \gamma} |\gamma|}$$

[Penington, Walter, Witteveen '22]

$\rightarrow$ Entanglement measure capturing genuinely tripartite information.

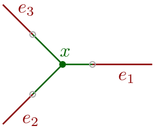
Can we explore the space of multipartite entanglement measures more systematically? at least for simple networks?

[w.i.p with Johann Chevrier, Luca Lionni, Michael Walter]

# RTNs with reduced randomness: local Haar-averaging

# Basic idea

Keep local entanglement structure at each vertex of the network fixed i.e. *average over local unitaries* instead of full unitary group.




$$\mathcal{H}_x = \mathcal{H}_{x,e_1} \otimes \mathcal{H}_{x,e_2} \otimes \mathcal{H}_{x,e_3}$$

Questions:

- Can such reduced randomness support area laws?
- If so, can we identify specific entanglement structures which do so?
- Can this framework produce richer entanglement spectra than in the standard case (non-flat spectra)?

# Local Haar-averaging

Suppose $x$ is a $q$-valent vertex, $q \geq 3$.

$$|T\rangle_x = \sum_{a_1, a_2 \cdots a_q} \underbrace{T_{a_1 a_2 \cdots a_q}}_{\text{fixed tensor}} |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_q\rangle$$

Average over $U(D)^{\otimes q} \subsetneq U(Dq)$:

$|\Psi\rangle_x \sim$ uniform random state in LU-orbit of $|T\rangle_x$, that is:

$$\mathbb{E}\left[|\Psi\rangle_{xx}\langle\Psi|^{\otimes n}\right] := \int_{U(D)^{\otimes q}} \mathrm{d}U \left[\left(\bigotimes_{c=1}^{q} U^{(c)}\right) \underbrace{|T\rangle_{xx}\langle T|}_{\text{seed state}} \left(\bigotimes_{c=1}^{q} U^{(c)}\right)^{\dagger}\right]^{\otimes n}$$

# Explicit evaluation of moments

Weingarten calculus allows to write: [Collins et al. '00s]

$$\mathbb{E}\left[|\Psi\rangle_{xx}\langle\Psi|^{\otimes n}\right]_{\{\mathbf{i}_s,\mathbf{j}_s\}_{s=1}^n} = \sum_{\boldsymbol{\sigma}} \underbrace{F_T(\boldsymbol{\sigma})}_{\text{state-dep. weight}} \underbrace{\mathcal{I}^{\boldsymbol{\sigma}}_{\{\mathbf{i}_s,\mathbf{j}_s\}_{s=1}^n}}_{\substack{\text{tensor structure} \\ \sim \text{colored diagram}}}$$

where $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_q) \in {S_n}^{\times q}$ etc. and

$$\mathcal{I}^{\boldsymbol{\sigma}}_{\{\mathbf{i}_s,\mathbf{j}_s\}_{s=1}^n} = \prod_{c=1}^{q} \prod_{s=1}^{n} \delta_{i_s^c, j^c_{\sigma_c(s)}}$$

$$F_T(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\tau}} \underbrace{\mathrm{Tr}_{\boldsymbol{\tau}}(\bar{T}, T)}_{\text{LU invariant}} \underbrace{\mathbf{W}^D(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1})}_{\substack{\text{Weingarten} \\ \text{functions}}}$$

Instead of one permutation per vertex, we have **two multiplets $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$**.

However, the Weingarten functions encourages $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ to be "close to each other".

# Asymptotics of the Weingarten function

$$D^n W^{(D)}(\sigma\tau^{-1}) = D^{-\overbrace{d(\sigma,\tau)}^{\text{Cayley distance}}} \underbrace{\mathrm{M}(\sigma\tau^{-1})}_{\substack{\text{Moebius function} \\ \text{(no definite sign)}}} \left(1+\mathcal{O}(1/D^2)\right)$$

For some entangled seed states, this allows to write:

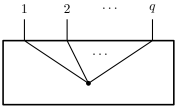
$$F_T(\boldsymbol{\sigma}) = \sum_{\boldsymbol{\tau}_1} D^{-\kappa(\mathcal{B}_\tau)-\sum_{c=1}^{q} d(\sigma_c,\tau_c)-\omega_T(\boldsymbol{\tau})} \left(\prod_{c=1}^{q} \mathrm{M}(\sigma_c\tau_c^{-1})\right)\left(1+\mathcal{O}(1/D^2)\right)$$

where

- $\kappa(\mathcal{B}_\tau) = n - \#\{\text{connected components of } \mathcal{B}_{\boldsymbol{\tau}}\}$ is minimal when $\tau_1 = \tau_2 = \ldots = \tau_q$;
- $\sum_{c=1}^{q} d(\sigma_c, \tau_c)$ is minimal when $\boldsymbol{\sigma} = \boldsymbol{\tau}$;
- $\omega_T(\boldsymbol{\tau})$ is a state-dependent contribution.

# $\omega_T(\boldsymbol{\tau})$ FOR SOME EXAMPLES OF SEED STATES

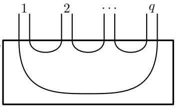
1. GHZ state:




$$|T\rangle_x = D^{(q-1)/2} \sum_{i=1}^{D} \bigotimes_{c=1}^{q} |i\rangle_c \quad \Rightarrow \quad \boxed{\omega_T(\boldsymbol{\tau}) = 0}$$

2. "Cyclic" state:

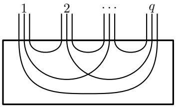
$|T\rangle_x = D^{q/4}$




$$\Rightarrow \quad \boxed{\omega_T(\boldsymbol{\tau}) = \underbrace{g_J(\mathcal{B}_{\boldsymbol{\tau}})}_{\text{genus of "jacket" } J=(12\cdots q)}}$$

3. "Complete graph" state:

$|T\rangle_x \propto$




$$\Rightarrow \quad \boxed{\omega_T(\boldsymbol{\tau}) = \omega_{\text{Gurau}}(\mathcal{B}_{\boldsymbol{\tau}})}$$

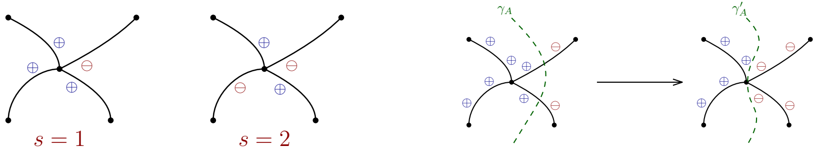
[R. Gurau's talk]

# EXAMPLE I: GHZ WITH $n = 2$ AND $q = 4$

$S_2 = \{\oplus, \ominus\}$. One can expicitly sum over $\boldsymbol{\tau}$, and derive a generalized spin model governed by energy:

$$\mathcal{E}(\{\boldsymbol{\sigma}_v\}) = \mathcal{E}_{\text{Ising}}(\{\boldsymbol{\sigma}_v\}) + \underbrace{2\nu_1 + \nu_2}_{\text{vertex defects}}$$

where $\nu_s := \#\{\textit{defects of type } s\}$.




Claim. Energy minimizers are not Ising configurations in general; as a result: we have $c_A \leq |\gamma_A|$ (with $c_A < |\gamma_A|$ for some networks) such that

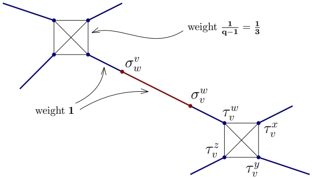
$$\mathbb{E}[\text{tr}(\rho_A^2)] = \mathcal{N}_{\text{g.s.}} \exp\left(-\ln(D) c_A\right) \left(1 + \mathcal{O}(1/D)\right) , \qquad \boxed{\mathbb{E}[S_2(\rho_A)] \approx c_A \ln(D)}$$

# Example II: complete graph seed state

A "spin" configuration is labelled by

$$s = \{\sigma_v^w, \sigma_w^v, \tau_v^w, \tau_w^v \,|\, (v, w) \in E\}$$

Leading order contributions minimize the Ising energy $\mathcal{E}_{\text{Ising}}(s)$ of a *refined network*




Conjecture. The Rényi entropy of a subregion $A$ is governed by the size of a minimal-cut $|\gamma_A|$

$$\mathbb{E}[\text{tr}(\rho_A^2)] = \mathcal{N}_{\text{g.s.}} \exp\left(-\ln(D)|\gamma_A|\right)(1 + \mathcal{O}(1/D)) \,, \qquad \boxed{\mathbb{E}[S_2(\rho_A)] \approx |\gamma_A| \ln(D)}$$

*To be checked:* absence of cancellation in the leading-order sector (which could arise due to non-positivity of Moebius function). Treacherous...

# Conclusion

- RTNs with reduced randomness, allowing for tunable entanglement structure at each vertex.
- Rényi entropy evaluation maps to generalized spin models: permutation associated to half-edges rather than edges, and energy contribution from internal structure of vertices.
- First examples with homogeneous choice across the network suggest that distinct choices of local entanglement structures affect the entanglement spectrum of the global state.
- In principle, the local entanglement structure could be chosen non-homogeneously across the network $\rightarrow$ large variety of effective behaviour can be expected, but likely hard to investigate in detail.
- How is the multipartite entanglement structure of the global state affected?

[Carrozza, Lionni + ... w.i.p.]