Free Cumulants For Random Tensors

Răzvan Gurău (IHP, 2024)

joint work with Benoît Collins and Luca Lionni

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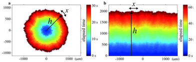
Random matrices

[Wishart '28, Wigner '55]

- Theory of strong interactions ['t Hooft, etc.]
- Random surfaces [David, Kazakov, Fröhlich, etc.]
- Growing interfaces fluctuations [Kardar, Parisi, Zhang, etc.]



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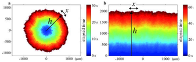

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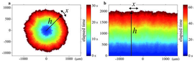

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How about tensors?

Random tensors

Random matrices (M_{ab}) generalize to random higher order tensors (T_{abc}) [Ambjørn Durhuus Jonsson '90, Sasakura '90, Boulatov '92, Ooguri '92, ...] and [2010: RG, Rivasseau, Oriti, Bonzom, Carrozza, Benedetti, Lionni, Tanasa, Ben Geloun, Ramgoolam, Dartois, Sasakura...]

- $1/N$ expansion (like random matrices)
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Lately – increased efforts to generalize "freeness" to tensors

- Identify the right objects at finite N, take the limit $N \to \infty$ and find their intrinsic defining properties
- Self contained formulation of the limit theory, without going through finite N first

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Strategy – mimic what works for matrices, start at finite N and take the limit

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- pairing of white and black labels $\to$ partition of $\{1, \ldots n, \bar{1}, \ldots \bar{n}\}$ as $\Pi(\sigma) = \big\{\{s, \overline{\sigma(s)}\}, \forall s\big\}$

$$(1\,3\,2) \to \big\{\{1,\bar{3}\},\{3,\bar{2}\},\{2,\bar{1}\}\big\}\,, \qquad (1)(2)(3) \to \big\{\{1,\bar{1}\},\{2,\bar{2}\},\{3,\bar{3}\}\big\}\,,$$

Invariant tensor probability measures

Basic building block $\rightarrow$ complex tensor T

N^D complex numbers $(T_{a^1,\dots a^D}\,,\ \bar{T}_{a^1,\dots a^D})$, $a^c = 1, \dots N$

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Probability measure for $(T_{a^1,\dots a^D}\ ,\ \bar{T}_{a^1,\dots a^D})$ with expectations invariant under local unitary transformations

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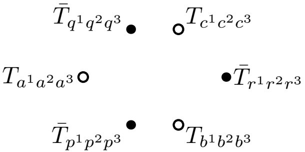

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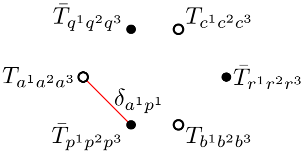

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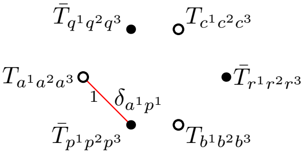

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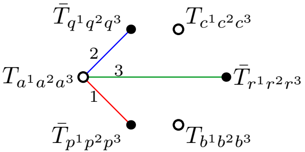

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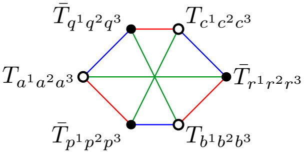

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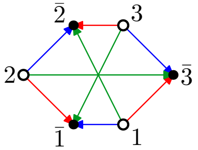
Invariant "traces" $\sum_{a^1,q^1} \delta_{a^1 q^1} \dots T_{a^1\dots a^D} \bar{T}_{q^1\dots q^D} \dots$ $\to$ colored graphs $\to$ D-tuples of permutations

$$\mathrm{Tr}_{\boldsymbol{\sigma}}(T,\bar{T}) = \sum_{i,j} \left(\prod_{s=1}^n T_{i_s^1 \dots i_s^D} \bar{T}_{j_{\bar{s}}^1 \dots j_{\bar{s}}^D} \right) \prod_{s=1}^n \prod_{c=1}^D \delta_{i_s^c j^c_{\overline{\sigma_c(s)}}}$$



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edges of color $c \to$ pairing $\{s, \overline{\sigma_c(s)}\}$
associated to the permutation σ_c

$$\sigma_1^{\text{red}} = (132) \qquad \{1,\bar{3}\}\{3,\bar{2}\}\{2,\bar{1}\}$$
$$\sigma_2^{\text{blue}} = (1)(2)(3) \qquad \{1,\bar{1}\}\{2,\bar{2}\}\{3,\bar{3}\}$$
$$\sigma_3^{\text{green}} = (123) \qquad \{1,\bar{2}\}\{2,\bar{3}\}\{3,\bar{1}\}$$

Lemma (Ben Geloun, Ramgoolam; Collins, RG, Lionni)

Denote $\boldsymbol{\sigma} = (\sigma_1, \dots \sigma_D), \sigma_c \in S(n)$. *For* $N > (n!)^{D-2}$, *the family:*

$$Tr_{\boldsymbol{\sigma}}(T, \bar{T}) = \sum_{i,j} \left(\prod_{s=1}^{n} T_{i_s^1 \dots i_s^D} \bar{T}_{j_{\bar{s}}^1 \dots j_{\bar{s}}^D} \right) \prod_{s=1}^{n} \prod_{c=1}^{D} \delta_{i_s^c j^c_{\overline{\sigma_c(s)}}} ,$$

up to relabeling $\boldsymbol{\sigma} \to \eta \boldsymbol{\sigma} \nu$ *is a basis in the space of homogeneous invariant polynomials of degree n in* T *and* $\bar{T}$.

Basis and decomposition

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Decomposition $\to$ averaging over $\mathbf{U} = U^{(1)} \otimes \dots U^{(D)}$:

$$f(T, \bar{T}) = \int d\mathbf{U} \, f(\mathbf{U}T, \bar{T}\mathbf{U}^\dagger)$$

$$\int dU \; U_{a_1 i_1} \dots U_{a_n i_n} \overline{U_{b_{\bar{1}} j_{\bar{1}}} \dots U_{b_{\bar{n}} j_{\bar{n}}}} = \sum_{\sigma, \tau \in S(n)} \prod_{s=1}^{n} \delta_{a_s b_{\overline{\tau(s)}}} \; \delta_{i_s j_{\overline{\sigma(s)}}} \quad \underbrace{W(\sigma\tau^{-1})}_{\text{Weingarten functions}}$$

Expectations and connected expectations

Random variables $x_1, \dots x_n, \dots$

$$\underbrace{\mathbb{E}[x_1, \dots x_n]}_{\text{expectation}} = \underbrace{\sum_{\pi \leq 1_n}}_{\text{Partitions of } n \text{ elements}} \prod_{B \in \pi} \underbrace{k[\{x_i, i \in B\}]}_{\text{connected expectation}}$$

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Partitions are lattice for the refinement order $\rightarrow$ Möbius inversion

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Multiplicative extensions $\mathbb{E}_\pi = \prod_{B \in \pi} \mathbb{E}[B]$ and $k_\pi = \prod_{B \in \pi} k[B]$

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Main Message

We identified large N **asymptotic moments** (not what you expect), **free cumulants** (not that simple) and a **lattice** (this one is fun!) such that **asymptotic moments cumulants relations** for random tensors hold.

Wishart matrices

Gaussian i.i.d. matrix entries:

$$\mathbb{E}[f(X,\bar{X})] = \int [dX d\bar{X}] \ f(X,\bar{X}) \ e^{-N(\sum_{a,b} X_{ab}\bar{X}_{ab})_{\nwarrow \mathrm{Tr}[XX^\dagger]}} \ ,$$

the only non zero connected expectation $k(X_{i^1 i^2}\bar{X}_{j^1 j^2}) = N^{-1}\delta_{i^1 j^1}\delta_{i^2 j^2}$

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But we are interested in other “connected” expectations:

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Φ “classical cumulants” defined via
moments cumulants relations in an appropriate lattice!

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$\Pi(\boldsymbol{\sigma})\le\Pi\le 1_{n,\bar{n}}$ lattice interval in the lattice of partitions $\to$ Möebius inversion works with the same Möebius function λ_Π:

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Equivalently, in terms of partitions among the connected components $\boldsymbol{\sigma}_1 \dots \boldsymbol{\sigma}_q$:

$$\mathbb{E}[\mathrm{Tr}_{\boldsymbol{\sigma}_1} \dots \mathrm{Tr}_{\boldsymbol{\sigma}_q}] = \sum_{\pi \leq 1_q} \prod_{B \in \pi} \Phi_{\bigcup_{j \in B} \boldsymbol{\sigma}_j}$$

Finite N free cumulants

The generating function of connected expectations is invariant:

$$W(J,\bar{J}) = \ln \mathbb{E}[e^{N^{D/2}\sum_a (\bar{J}_{a^1\dots a^D}\bar{T}_{a^1\dots a^D} + J_{a^1\dots a^D}T_{a^1\dots a^D})}] = \sum_{\boldsymbol{\sigma}} \mathrm{Tr}_{\boldsymbol{\sigma}}(J,\bar{J})\,\mathcal{K}_{\boldsymbol{\sigma}}[T,\bar{T}]$$

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Theorem

The finite N free cumulants write in terms of the classical cumulants as:

$$\mathcal{K}_{\boldsymbol{\sigma}}[T,\bar{T}] = N^{nD}\sum_{\boldsymbol{\tau}}\sum_{\Pi(\boldsymbol{\tau})\vee\Pi(\boldsymbol{\sigma})\leq\Pi''}\lambda_{\Pi''}\underbrace{\Big(\sum_{\Pi(\boldsymbol{\tau})\leq\Pi'\leq\Pi''}\Phi_{\Pi',\boldsymbol{\tau}}(T,\bar{T})\Big)}_{\mathbb{E}_{\Pi'',\boldsymbol{\tau}}(T,\bar{T})}\prod_{B\in\Pi''}\prod_{c=1}^{D}W(\sigma_{c|_B}\tau_{c|_B}^{-1})$$

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Proof $\rightarrow \ln(e^x) = \ln(1+(e^x-1)) = \sum_{q\ge 1}\frac{(-1)^q}{q}(e^x-1)^q\dots$ hence:

$$\ln \mathbb{E}[e^{f(T,\bar{T})}] = \sum_{n\ge 1}\frac{1}{n!}\sum_{\pi\le 1_n}\lambda_\pi\prod_{B\in\pi}\mathbb{E}[\int d\mathbf{U}\, f(\mathbf{U}T,\bar{T}\mathbf{U}^\dagger)^{|B|}]$$

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- large N factorization $\{\{1\},\{2\},\dots\{q\}\}$ dominates $\Leftrightarrow$ $r(\boldsymbol{\sigma})$ strictly sub additive

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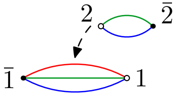
We choose the same scaling $r(\boldsymbol{\sigma})$ as in the Gaussian case[a] which we conjecture to be strictly sub additive, but we make no assumptions on $\varphi_{\boldsymbol{\sigma}}(t,\bar{t})$.

[a] $[d\bar{T}dT]\exp\{-N^{D-1}\sum T_{a^1\dots a^D}\bar{T}_{a^1\dots a^D}\}$, $\quad r(\boldsymbol{\sigma}) = n - \min_{\eta\in S_n, \Pi(\boldsymbol{\sigma})\vee\Pi(\eta)=1_{n,\bar{n}}} \sum_{c=1}^{D} |\sigma_c\eta^{-1}|$.

The melon strikes back / Return of the melon / Revenge of the melon / The melon awakens / The rise of Melon



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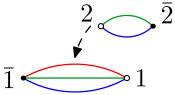

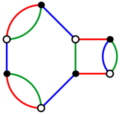

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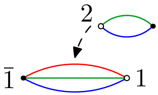

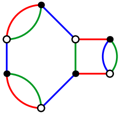

$$\begin{cases}\sigma_1 & = (1) \\ \sigma_2 & = (1) \\ \sigma_3 & = (1)\end{cases} \to \begin{cases}\sigma_1 & = (1\,2) \\ \sigma_2 & = (1)(2) \\ \sigma_3 & = (1)(2)\end{cases}$$

Well labelled melon at step n:

- insert n in a cycle in one σ_c
- append fixed point (n) to $\sigma_{c', c' \neq c}$

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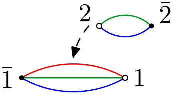

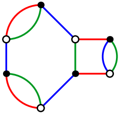

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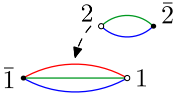
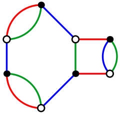
Theorem (RG)

We have $r(\boldsymbol{\sigma}) = D - (D-1)|\Pi(\boldsymbol{\sigma})| - \Omega(\boldsymbol{\sigma})$ *where* $|\Pi(\boldsymbol{\sigma})|$ *is the number of connected components of* $\boldsymbol{\sigma}$ *and* $\Omega(\boldsymbol{\sigma}) \geq 0$*. Furthermore,* $\Omega(\boldsymbol{\sigma}) = 0$ *if and only if* $\boldsymbol{\sigma}$ *is* **melonic**.

The leading invariants are **connected, melonic** *and at fixed* $|\Pi(\boldsymbol{\sigma})|$ *the leading invariants are melonic.*

The melon strikes back / Return of the melon / Revenge of the melon / The melon awakens / The rise of Melon





$$\begin{cases} \sigma_1 &= (1) \\ \sigma_2 &= (1) \\ \sigma_3 &= (1) \end{cases} \to \begin{cases} \sigma_1 &= (1\,2) \\ \sigma_2 &= (1)(2) \\ \sigma_3 &= (1)(2) \end{cases}$$

Well labelled melon at step n:

- insert n in a cycle in one σ_c
- append fixed point (n) to $\sigma_{c',c'\neq c}$

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For $D = 2$ all the invariants are melonic (bi colored cycles)!

A flip on a well labelled melon $\boldsymbol{\sigma}-$ split a cycle of one σ_c into two:

$$(i_1 \dots i_p i_{p+1} \dots i_q i_{q+1} \dots i_l) \to (i_1 \dots i_p i_{q+1} \dots i_l)(i_{p+1} \dots i_q)$$

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The flip partial order

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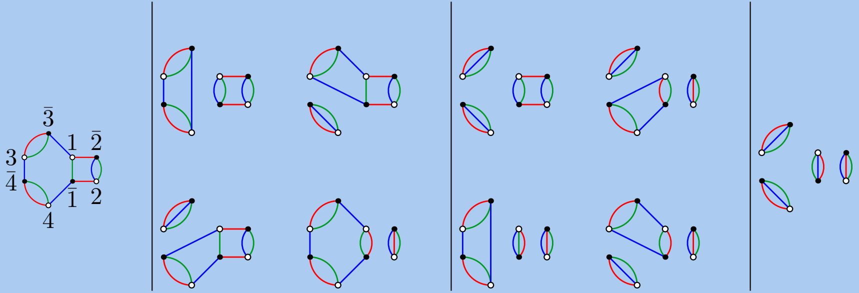
Cut two edges on the same cycle of some σ_c; reconnect respecting coloring



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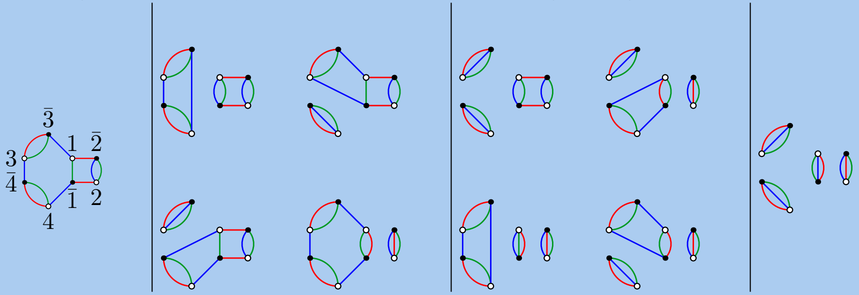

Flip partial order: $\boldsymbol{\tau} \preceq \boldsymbol{\sigma}$ if $\boldsymbol{\tau}$ can be obtained from $\boldsymbol{\sigma}$ by a sequence of flips ($\boldsymbol{\tau}$ is melonic)

Melonic cumulants

τ_c can be obtained from σ_c by a sequence of flips $\Leftrightarrow$ τ_c is *non-crossing* on σ_c

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Theorem (Collins, RG, Lionni)

For **well labeled melonic, connected** *invariants $\boldsymbol{\sigma}$ in $D \geq 3$ the free cumulants are:*

$$\kappa_{\boldsymbol{\sigma}}(t,\bar{t}) = \lim_{N\to\infty} \frac{\mathcal{K}_{\boldsymbol{\sigma}}}{N^{r(\boldsymbol{\sigma})=1}} = \sum_{\boldsymbol{\tau}\preceq\boldsymbol{\sigma}} \mathrm{M}(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1}) \prod_{i=1}^{|\Pi(\boldsymbol{\tau})|} \varphi_{\boldsymbol{\tau}_i}(t,\bar{t}) \ , \quad \varphi_{\boldsymbol{\sigma}}(t,\bar{t}) = \sum_{\boldsymbol{\tau}\preceq\boldsymbol{\sigma}} \prod_{i=1}^{|\Pi(\boldsymbol{\tau})|} \kappa_{\boldsymbol{\tau}_i}(t,\bar{t}) \ ,$$

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$$\mathcal{K}_{\boldsymbol{\sigma}} = N^{nD} \sum_{\boldsymbol{\tau}} \sum_{\Pi(\boldsymbol{\tau})\vee\Pi(\boldsymbol{\sigma})\leq\Pi''} \lambda_{\Pi''} \Big(\sum_{\Pi(\boldsymbol{\tau})\leq\Pi'\leq\Pi''} \Phi_{\Pi',\boldsymbol{\tau}} \Big) \prod_{B\in\Pi''} \prod_{c=1}^{D} W(\sigma_{c|_B}\tau_{c|_B}^{-1})$$

$\boldsymbol{\sigma}$ connected $\Rightarrow \Pi(\boldsymbol{\sigma}) = 1_{n,\bar{n}} \Rightarrow \Pi'' = 1_{n,\bar{n}}$, and $\lambda_{1_{n,\bar{n}}} = 1$

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$W(\nu) \sim N^{-n-|\nu|} M(\nu)$, and $\Phi_{\Pi',\tau} \sim N^{\sum_{B'\in\Pi'} r(\boldsymbol{\tau}|_{B'})} \prod_{B'} \varphi_{\boldsymbol{\tau}|_{B'}} \Rightarrow \Pi' = \Pi(\boldsymbol{\tau}), \boldsymbol{\tau} \preceq \boldsymbol{\sigma}$

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$$M(\nu) = \underbrace{\prod_{p\geq 1} \Bigg[(-1)^{p-1} \frac{1}{p} \underbrace{\binom{2p-2}{p-1}}_{\text{Catalan number}} \Bigg]^{\overbrace{d_p(\nu)}^{\text{number of cycles of length } p \text{ of } \nu}}}_{\text{Möebius function for lattice of non-crossing partitions}} \ , \quad M(\boldsymbol{\nu}) = \prod_c M(\nu_c)$$

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What is an infinite random tensor?