Random tensors and Related Topics
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Entanglement detection via tensor norms

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M.A. Jivulescu, Cécilia Lancien, Ion Nechita *Multipartite entanglement detection via projective tensor norms*, Annales Henri-Poincare, 2022

- Metric interpretation for known entanglement criteria;
- Tensor norms in Banach spaces;
- Entanglement testers;
- Entanglement criteria and relations between them.

Metric interpretation for quantum states

Metric interpretation for quantum states

Pure q. states: $|\psi\rangle$ unit vector norms in Hilbert space $\mathcal{H} = \mathbb{C}^d$;

$$\text{quantum states} = \text{PSD}_d \cap \{\text{ Tr} = 1\}$$

Quantum states from metric point of view:

- pure multipartite states: for each space $\ell_2^{d_i} := (\mathbb{C}^{d_i}, \|\cdot\|_2)$
- mixed multipartite states: $S_1^d := (\mathcal{M}^d(\mathbb{C}), \|\cdot\|_1)$, $\|X\|_1 = Tr\sqrt{X^*X}$
- quantum states: $\{X \in \mathcal{M}_d^{sa,+}(\mathbb{C}) : \text{Tr}X = \|X\|_{S_1^d} = 1\}$

$\{$ Quantum states $\mathcal{H}_1 \otimes \mathcal{H}_2\} = PSD_{d_1 d_2} \cap \{\text{ Tr} = 1\}$

- **separable pure states**(or product states): $|\phi\rangle = |\varphi\rangle \otimes |\chi\rangle$; otherwise, the state is called **entangled**.
- **separable mixed states**: $\rho = \sum_{i=1}^{L} p_i \rho_i^1 \otimes \rho_i^2$ (i.e. convex combination of product states); otherwise, the state is called **entangled**.

- deciding if a state is separable or entangled is a problem of NP complexity[Gharibian, QCI, 2010];
- there are criteria of separability necessary and sufficient, but not practical (i.e. redefinition of separability condition)
- criteria easy to compute, but only necessary, or sufficient
- the most known/used:
  - Positive partial transposition criterion (PPT)[Peres, Horodecki, '96]
  - realignment criterion (RLN) [Chen, Wu2003]

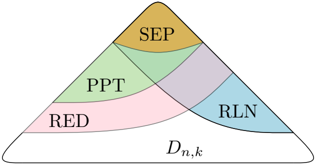
- **Realignment criterion[Chen, Wu2003]:**

  If ρ is a bipartite separable state in $\mathcal{M}_d(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$, then
  $$\|\rho^R\|_{S_1^{d^2}} \leq 1,$$
  where ρ^R is given by $\rho^R_{ij,kl} = \rho_{ik,jl}$

- operational and simple to compute
- Both PPT and RLN detect all pure entangled states
- RLN is not equiv/weaker/stronger to PPT, but complementary.

Entanglement criteria classification[1]

- **equivalent**: C1 and C2 detects the same states;
- **complementary**: C1 can detects states not detected by C2 and vice versa;
- C1 is **stronger** than C2: can detect all states that are detected by C2 and at least one more;
- C1 is **weaker** than C2: all states detected by C1 are also detected by C2 and C2 can detect at least one more;



[1] Kiara Hansenne, Quantum Entanglement, A study of recent separability criteria, 2020

- correlation matrix criterion [de Vicente, 2008]
- covariance matrix criterion[Guhne and all, 2007/2008]
- LWFL family of criteria[Li and all, 2014]
- criterion based on SIC-POVM[Shang and all, 2018]
- SSC family of criteria[Sabricki and all, 2018]
- entanglement criteria for classes of (N,M)-POVM [K. Siudzińska 2022]

Common features:

- express conditions in terms of trace norms: ρ sep $\rightarrow \|\mathcal{T}(\rho)\|_1 \leq 1$.
- mostly are formulated for bipartite case

Entanglement criteria using tensor norms

Definition

Consider m Banach spaces $A_1, \ldots, A_m$. For a tensor $x \in A_1 \otimes \cdots \otimes A_m$, we define its **projective tensor norm**

$$\|x\|_\pi := \inf \left\{ \sum_{k=1}^{r} \|a_k^1\| \cdots \|a_k^m\| \ : \ r \in \mathbb{N}, \ a_k^i \in A_i, \ x = \sum_{k=1}^{r} a_k^1 \otimes \cdots \otimes a_k^m \right\}, \tag{1}$$

and its **injective tensor norm**

$$\|x\|_\epsilon := \sup \left\{ | \langle \alpha^1 \otimes \cdots \otimes \alpha^m | x \rangle | \ : \ \alpha^i \in A_i^*, \ \|\alpha^i\| \leq 1 \right\}. \tag{2}$$

- for simple tensors: $\|a_1 \otimes \cdots \otimes a_m\|_\pi = \|a_1 \otimes \cdots \otimes a_m\|_\epsilon = \|a_1\| \cdots \|a_m\|$,
- $\forall \ x \in A_1 \otimes \cdots \otimes A_m, \quad \|x\|_\epsilon \leq \|x\| \leq \|x\|_\pi$.
- are dual to each other
- basic examples: $\| \cdot \|_{S_1^d} = \| \cdot \|_{\ell_2^d \otimes_\pi \ell_2^d}$ and $\| \cdot \|_{S_\infty^d} = \| \cdot \|_{\ell_2^d \otimes_\epsilon \ell_2^d}$

Relevance of tensor norms in QIT

Observations:

- computing tensor norms (≥ 3) factors is NP-hard[Hendrick2010, Hillar2013]
- concrete computations have been done only for some specific examples[Friedland 2017]
- numerically approaches are known: based on tensor ranks computation[Bruzda2022], SOCP[Darksen2017], SDP algorithms for tensors with random asymmetric component[Kivva2021], other algorithms[Fitter2022].

Applicability:

- entanglement detection[2]
- (in)compatibility of quantum measurements[3]

[2] [Fitter2022],[Jivulescu2020]
[3] [Bluhm2022-1, Bluhm2022-2], [Faedi2022].

Proposition

A **multipartite pure quantum state** $\psi \in \mathbb{C}^{d_1} \otimes \cdots \otimes \mathbb{C}^{d_m}$, $\|\psi\|_2 = 1$,

$$\psi \text{ is separable iff } \|\psi\|_\epsilon = \|\psi\|_\pi = 1$$

- **Geometric measure of entanglement**[4]

$$G(\psi) := -\log \sup_{\varphi_i \in H_i,\ \|\varphi_i\|=1} \left\{ |\langle \varphi_1 \otimes \cdots \otimes \varphi_m | \psi \rangle|^2 \right\} = -2 \log \|\psi\|_\epsilon .$$

Theorem[5]

For a **multipartite mixed quantum state** $\rho \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_m}(\mathbb{C})$, $\rho \geq 0$, $\mathrm{Tr}\,\rho = 1$, the following assertions are equivalent:

1. ρ is separable,
2. $\|\rho\|_{S_{1,sa}^{d_1} \otimes_\pi \cdots \otimes_\pi S_{1,sa}^{d_m}} = 1$,
3. $\|\rho\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m}} = 1$.

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[4] Shimony 1995, Wei, Goldbart 2003

[5] Rudolf 2000 si David Perez-Garcia 2004

Entanglement testers[6]

[6] M.A. Jivulescu, Cécilia Lancien, Ion Nechita *Multipartite entanglement detection via projective tensor norms*, Annales Henri-Poincare, 2022

Definition

To a n-tuple of matrices $(E_1, \ldots, E_n) \in (\mathcal{M}_d(\mathbb{C}))^n$, we associate the linear map

$$\mathcal{E} : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \sum_{k=1}^{n} \mathrm{Tr}(E_k^* X) \, |k\rangle \in \mathbb{C}^n,$$

where $\{|k\rangle\}_{k=1}^n$ is some fixed orthonormal basis of $\mathbb{C}^n$.

The map $\mathcal{E}$ is called **entanglement tester** if $\|\mathcal{E}\|_{S_1^d \to \ell_2^n} = 1$.

- use $\mathcal{E}$ as local contractions
- reduce the problem of multipartite mixed states to multipartite pure states (simpler, commutative)

Corollary

Let $E_i = \{E_{i;k}\}_{k=1}^{n_i}$, $1 \leq i \leq m$, be m sets of operators $\mathcal{E}_1, \dots, \mathcal{E}_m$ the corresponding linear maps. Then, for any $X \in \mathcal{M}_{d_1}(\mathbb{C}) \otimes \cdots \otimes \mathcal{M}_{d_m}(\mathbb{C})$:

$$\|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(X)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} \leq \|\mathcal{E}_1\|_{S_1^{d_1} \to \ell_2^{n_1}} \cdots \|\mathcal{E}_m\|_{S_1^{d_m} \to \ell_2^{n_m}} \|X\|_{S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m}}$$

In particular, if the $\mathcal{E}_i$'s are testers,

$$\rho \text{ separable} \implies \|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} \leq 1.$$

Reciprocally, we have the following **entanglement criterion**:

$$\|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} > 1 \implies \rho \text{ is entangled.}$$

Reduction of difficulty: from $2m$ factors to m factors in evaluation of $S_1^{d_1} \otimes_\pi \cdots \otimes_\pi S_1^{d_m} \cong (\ell_2^{d_1} \otimes_\pi \ell_2^{d_1}) \otimes_\pi \dots \otimes_\pi (\ell_2^{d_m} \otimes_\pi \ell_2^{d_m})$) to that of $\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}$.

Entanglement tester and related operators

For a given set of operators $E = \{E_k\}_{k=1}^{n}$ and the corresponding tester $\mathcal{E}$:

- test operator $T_E := \sum_{k=1}^{n} E_k \otimes E_k^*$
- $\mathcal{T}_E$-the c.p.map having E_k as Kraus operators: $\mathcal{T}_E(X) = \sum_{k=1}^{n} E_k X E_k^*$
- Choi operator associated to $\mathcal{T}_E$:

$$\Theta_E = \sum_{k=1}^{n} |e_k\rangle \langle e_k| \, , \text{ where } |e_k\rangle = \sum_{i,j=1}^{d} < i|E_k|j > |ij\rangle$$

- The set of test operators on $\mathbb{C}^d \otimes \mathbb{C}^d$ is

$$\{\Theta_E^\Gamma F, \Theta \geq 0, \|\Theta_E\|_{S^d_{\infty,sa} \otimes_\epsilon S^d_{\infty,sa}} = 1\}$$

Definition

Two testers $\mathcal{E}, \mathcal{F} : S_1^d \to \ell_2^n$ are called **equivalent** if there exists a unitary operator $U \in \mathcal{U}(n)$ such that

$$\mathcal{F}(X) = U\mathcal{E}(X), \forall X \in \mathcal{M}_d(\mathbb{C})$$

Remark

Two testers $\mathcal{E}, \mathcal{F} : S_1^d \to \ell_2^n$ are called equivalent if and only if they have the same test operator

$$T_E = T_F$$

Realignment criterion: $\|\rho^R\|_{S_1^{d^2}} > 1 \Rightarrow \rho$ entanglement

Reformulation:

- matrices $\{R_{ij}\}_{i,j=1}^d = \{|i\rangle\langle j|\}_{i,j=1}^d$, where $|i\rangle \in \mathbb{C}^d$ orthonormal basis
- the map $\mathcal{R} = id : X \in \mathcal{M}_d(\mathbb{C}) \mapsto \sum_{i,j=1}^d \langle i| X |j\rangle |ij\rangle \in \mathbb{C}^{d^2}$
- $\mathcal{R}$- entanglement tester: $\|\mathcal{R}\|_{S_1^d \to \ell_2^{d^2} \cong S_2^d} = 1$
- test operator: $T_R = F := \sum_{i,j=1}^d |i\rangle\langle j| \otimes |j\rangle\langle i|$ (flip operator)
- $\rho^R = [\mathcal{R} \otimes \mathcal{R}](\rho)$ and Realig. Crit. corresponds to $\mathcal{R} \otimes \mathcal{R}$ tester
- we need now to compute $\|\mathcal{R} \otimes \mathcal{R}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}}$ instead of $\|\rho^R\|_{S_1^{d^2}}$
- generalize to multiparite settings using as tester $\mathcal{R}^{\otimes m}$

Correlation matrix criterion(SSC)

- Let $G=\{G_k\}_{k=1}^{d^2}$ is the canonical orthonormal basis: $G_1 = I/\sqrt{d}$ and the others traceless
- **SSC criteria**[7]: for C the correlation matrix $C_{kl} = Tr(\rho G_k \otimes G_l)$ and $D_x = diag\{x, 1, \ldots, 1\}$ $(x, y \geq 0$ fixed) it holds that

$$\rho \text{ separable} \Rightarrow \|D_x C D_y\|_1 \leq \sqrt{\tfrac{d-1+x^2}{d}}\sqrt{\tfrac{d-1+y^2}{d}}$$

- Realig. criterion for $x=y=1$; Correlation criterion(J.de Vicente, 2007) for $x=y=0$; Li, Wang, Fei(PRA 2008) criterion for $x=y=\sqrt{2/d}$ and ESIC criterion for $x=y=\sqrt{d+1}$,.
- define $\widetilde{\mathcal{G}}_x : X \mapsto x\mathrm{Tr}(G_1^* X)\,|1\rangle + \sum_{k=2}^{d^2} \mathrm{Tr}(G_k^* X)\,|k\rangle$ and entanglement tester

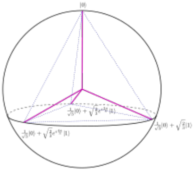
$$\mathcal{G}_x := \left(\frac{d}{d-1+x^2}\right)^{1/2} \widetilde{\mathcal{G}}_x$$

- SSC criteria corresponds to $\mathcal{G}_x \otimes \mathcal{G}_y$ tester: $\rho^G = [\mathcal{G}_x \otimes \mathcal{G}_y](\rho)$.

[7] Sarbicki, Scala, Cruscinski, PRA, 2020

ESIC-POVM criterion

- $\{|x_k\rangle\}_{k=1}^{d^2}$ finite subset of the unit sphere of $\mathbb{C}^d$ such that $|\langle x_i|x_j\rangle|^2 = \frac{d\delta_{ij}+1}{d+1}$
- **SIC-POVM:** symmetric family of rank-1 operators: $\{\Pi_k = \frac{1}{d}|x_k\rangle\langle x_k|\}$
- existence proven for $d = 1, \ldots, 16, 19, 24, 35, 48, \ldots$



- **ESIC criterion**: ρ separable state $\Rightarrow \|\rho^S\|_{S_1^{d^2}} \leq 1$, $[\rho^S]_{ij} = Tr[\rho\Pi_i^1 \otimes \Pi_j^2]$

SIC-POVM case

Reformulation of ESIC criterion:

- matrices $\{S_k\}_{k=1}^{d^2} = \{\sigma |x_k\rangle \langle x_k|\}_{k=1}^{d^2}$, $\sigma = \sqrt{(d+1)/(2d)}$
- entanglement tester: $\mathcal{S} : X \mapsto \sigma \sum_{k=1}^{d^2} \langle x_k| X |x_k\rangle |k\rangle$.
- $\rho^S = [\mathcal{S} \otimes \mathcal{S}](\rho)$
- SIC-POVM criterion corresponds to $\mathcal{S} \otimes \mathcal{S}$ tester
- test operator: $T_S = \frac{I+F}{2}$

Remark: Conjecture[8]: if $\|\rho^R\|_{S_1^{d^2}} > 1$, then $\|\rho^S\|_{S_1^{d^2}} > 1$

[8] Shang all, 2018

- A tester $\mathcal{E} : S_1^d \to \ell_2^n$ is called **symmetric** if its test operator $T_E := \sum_{k=1}^{n} E_k \otimes E_k^*$ can be written as $T_E = \alpha F + \beta I$.
- **Fact**:[9] Let $\{E_k\}_{k=1}^{d^2}$ be a basis of operators on $\mathbb{C}^d$. Then, the following statements are equivalent
  - $\sum_{k=1}^{d^2} E_k \otimes E_k^* = \alpha F + \beta I$,
  - $\mathrm{Tr}(E_k^* E_l) = \alpha \delta_{kl} + \gamma \mathrm{Tr}(E_k^*)\mathrm{Tr}(E_l) \quad \forall\ 1 \leq k, l \leq d^2$

  In this case, we have $\alpha > 0$, $\alpha + d\beta > 0$ and $\gamma = \beta/(\alpha + d\beta)$.

  They are called conical 2-designs.
- Realig case: $\alpha = 1, \beta = \gamma = 0$ and $T_R = F$
- ESIC POVM case $\alpha = \beta = 1/2, \gamma = 1/(d+1)$ and $T_S = (I + F)/2$.

[9] Appleby&all 2013

Entanglement detection by symmetric testers

- For $\mathcal{E} : S_1^d \to \ell_2^n$ a symmetric tester with corresponding parameters (α, β), then, for any bipartite unit vector $\varphi \in \mathbb{C}^d \otimes \mathbb{C}^d$ with Schmidt decomposition $|\varphi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |e_i f_i\rangle$, we have $\left\| \mathcal{E}^{\otimes 2}(|\varphi\rangle\langle\varphi|) \right\|_1 = \alpha + \beta + 2\alpha \sum_{i<j} \sqrt{\lambda_i \lambda_j}$.
- In Realig and SIC POVM case we have a necessary and sufficient condition for separability of bipartite pure states: $\|\mathcal{E}^{\otimes 2} |\varphi\rangle\langle\varphi| \|_1 \leq 1$
- $\left\| \mathcal{R}^{\otimes 2}(|\varphi\rangle\langle\varphi|) \right\|_1 = \| \, |\varphi\rangle\langle\varphi| \, \|_{S_1^d \otimes_\pi S_1^d}$[10]
- Moreover,[11] for any pure state φ, we have

$$\|\mathcal{S}^{\otimes 2}(|\varphi\rangle\langle\varphi|)\|_1 = \frac{\left\| \mathcal{R}^{\otimes 2}(|\varphi\rangle\langle\varphi|) \right\|_1 + 1}{2}$$

[10] Carlos Palazuelos, 2014

[11] proves the conjectured equality in Shang & all 2018

Isotropic states

Goal: to determine when the realignment and SIC POVM testers detect the entanglement of isotropic:

$$\tau_\mu := \mu|\psi\rangle\langle\psi| + (1-\mu)\frac{I}{d^2}\,,\ 0 \le \mu \le 1,$$

We get:

$$\mathcal{R}^{\otimes 2}(\tau_\mu) = \frac{1}{d}\left(\mu I + (1-\mu)|\psi\rangle\langle\psi|\right),$$
$$\mathcal{S}^{\otimes 2}(\tau_\mu) = \frac{1}{2d}\left(\mu I + \frac{d+1-\mu}{d^2}J\right).$$

Hence,

$$\left\|\mathcal{R}^{\otimes 2}(\tau_\mu)\right\|_1 > 1 \iff \left\|\mathcal{S}^{\otimes 2}(\tau_\mu)\right\|_1 > 1 \iff \mu > \frac{1}{d+1}.$$

Conclusion: both the realignment and the SIC POVM maps detect all entangled isotropic states.

Goal: to determine when the realignment and SIC POVM testers detect the entanglement of Werner states:

$$\sigma_\mu := \mu\frac{I+F}{d(d+1)} + (1-\mu)\frac{I-F}{d(d-1)}, \; 0 \leq \mu \leq 1.$$

Here, we can see that

$$\left\|\mathcal{R}^{\otimes 2}(\sigma_\mu)\right\|_1 > 1 \iff \left\|\mathcal{S}^{\otimes 2}(\sigma_\mu)\right\|_1 > 1 \iff \mu < \frac{1}{d}.$$

Fact: σ_μ entangled iff $\mu < 1/2$[Werner89].

So as soon as $d > 2$, both the realignment and the SIC POVM maps do not detect all entangled Werner states (and they perform increasingly poorly as d grows).

Question: there is any relation between the norms of the realignment and SIC POVM maps?

$$\left\|\mathcal{S}^{\otimes 2}(\tau_\mu)\right\|_1 = \frac{\left\|\mathcal{R}^{\otimes 2}(\tau_\mu)\right\|_1 + 1}{2} \text{ and } \left\|\mathcal{S}^{\otimes 2}(\sigma_\mu)\right\|_1 = \frac{\left\|\mathcal{R}^{\otimes 2}(\sigma_\mu)\right\|_1 + 1}{2}$$

Realig versus SIC using entanglement testers

Realig versus SIC POVM

Reformulation of Shang conjecture: Given an entangled state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, if its entanglement is detected by the matrix unit tester $\mathcal{R} : S_1^d \to \ell_2^{d^2}$, then it is necessarily detected by the SIC POVM tester $\mathcal{S} : S_1^d \to \ell_2^{d^2}$ as well, i.e.

$$\|\mathcal{R}^{\otimes 2}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} > 1 \Longrightarrow \|\mathcal{S}^{\otimes 2}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} > 1. \tag{3}$$

Analytical proof of the result based on:

Theorem

For any quantum state ρ on $\mathbb{C}^d \otimes \mathbb{C}^d$, we have

$$\|\mathcal{S}^{\otimes 2}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} \geq \frac{\|\mathcal{R}^{\otimes 2}(\rho)\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} + 1}{2}. \tag{4}$$

Comments:

- Ineguality (4) is an equality for several classes of states such as
  - pure states;
  - isotropic states: $\rho = p\omega_d + (1-p)I/d^2, 0 \leq p \leq 1$
  - Werner states: $\rho = q\frac{I+F}{d(d+1)} + (1-q)\frac{I-F}{d(d-1)}, 0 \leq q \leq 1$
  - product state $\rho = \rho_1 \otimes \rho_2$, with the same purity $Tr(\rho_1^2) = Tr(\rho_2^2)$;
- In general, the inequality is not saturated

Lemma

Let $\{a_i\}$, $\{b_n\}$ be two orthonormal bases of $\mathbb{C}^n$. For $\gamma_i \in \mathbb{C}$ such that $|\gamma_i| \geq 1$ for all $1 \leq i \leq n$, define the matrix $S := \sum_{i=1}^n \gamma_i |a_i\rangle \langle b_i|$. Then:

$$\|SXS^*\|_1 \geq \|X\|_1 + \sum_{i=1}^n (|\gamma_i|^2 - 1) \langle b_i| X |b_i\rangle, \forall X \in \mathcal{M}_n(\mathbb{C})$$

Idea of proof:

- S is invertible and $S^{-1} = \sum_{i=1}^n \gamma_i^{-1} |b_i\rangle \langle a_i|$.
- denote $Y := SXS^*$, so the ineq. becomes

$$\|Y\|_1 \geq \|S^{-1}Y(S^*)^{-1}\|_1 + \sum_{i=1}^n \left(1 - |\gamma_i|^{-2}\right) \langle a_i| Y |a_i\rangle. \quad (5)$$

- relation equivalent to contractivity of the map $\Phi : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_{2n}(\mathbb{C})$ is given by

$$\Phi(Y) = (S^{-1}Y(S^*)^{-1}) \oplus \left(\bigoplus_{i=1}^{n} (1 - |\gamma_i|^{-2}) \langle a_i| Y |a_i\rangle \right).$$

- show that Φ is a quantum channel
- define

$$K = \begin{bmatrix} S^{-1} \\ 0_n \end{bmatrix} \quad \text{and} \quad L_i = \begin{bmatrix} 0_n \\ \sqrt{1 - |\gamma_i|^{-2}}\, |i\rangle \langle a_i| \end{bmatrix},\ 1 \le i \le n.$$

- Φ complet positivite map as

$\Phi(X) = KXK^* + \sum_{i=1}^{n} L_i X L_i^*$

- Φ is trace preserving map as

$$K^*K + \sum_{i=1}^{n} L_i^* L_i = (S^{-1})^* S^{-1} + \sum_{i=1}^{n} (1 - |\gamma_i|^{-2}) |a_i\rangle \langle a_i| = \sum_{i=1}^{n} |a_i\rangle \langle a_i| = I_n.$$

Want to compare $\|\mathcal{G}_x^{\otimes 2}(\rho)\|_{\ell_2^{d^2}\otimes_\pi \ell_2^{d^2}}$ with $\|\mathcal{G}_y^{\otimes 2}(\rho)\|_{\ell_2^{d^2}\otimes_\pi \ell_2^{d^2}}$

Important observations:

- Given the set $\{E_k\}_{k=1}^n$, then the matrix of the map $\mathcal{E}$ is $\hat{E} = \sum_{k=1}^{n} |k\rangle \langle E_k|$
- $\|\mathcal{E}^{\otimes 2}(\rho)\|_{\ell_2^{d^2}\otimes_\pi \ell_2^{d^2}} = \|\hat{E}X\hat{E}^*\|_1$, where $X = \rho^R = \mathcal{R}^{\otimes 2}(\rho)$
- apply Lemma for $\hat{G}_x = \sqrt{\frac{d}{d-1+x^2}}\,\hat{\hat{G}}_x$

We have that

$$\|\hat{G}_y \rho^R \hat{G}_y^*\|_1 \geq (1-\lambda)\|\hat{G}_x \rho^R \hat{G}_x^*\|_1 + \lambda,$$

where $\lambda := \frac{y^2-x^2}{d-1+y^2} \in [0,1]$.

The case $x = 1$ and $y = \sqrt{d+1}$ (i.e. $\lambda = 1/2$) proves the conjecture!

Main idea: given any entangled bipartite state, there always exist testers detecting its entanglement?

Theorem

Let ρ be an entangled state on $\mathbb{C}^d \otimes \mathbb{C}^d$. Then, there exists a tester $\mathcal{E} : S_1^d \to \ell_2^{d^2}$ such that

$$\left\| \mathcal{E}^\sharp \otimes \mathcal{E}\left(\tilde{\rho}\right) \right\|_{\ell_2^{d^2} \otimes_\pi \ell_2^{d^2}} > 1, \;\; \tilde{\rho} = F \rho^\Gamma ,$$

$\mathcal{E}^\sharp : S_1^d \to \ell_2^{d^2}$ is the tester whose operators are the adjoints of those of $\mathcal{E}$.

- $\|\rho\|_{S_1^d \otimes_\pi S_1^d} = \|\tilde{\rho}\|_{S_1^d \otimes_\pi S_1^d}$
- test operator $T = \Phi^\Gamma F$, where Φ is an entanglement witness: $\langle \Phi, \rho \rangle > 1$ and $\|\Phi\|_{S_\infty^d \otimes_\varepsilon S_\infty^d} = 1$

Entanglement testers in the multipartite setting

Main result: φ entangled $\iff \|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\|_{(\ell_2^{d^2})^{\otimes_\pi m}} > 1$.

Ingredients of proof:

- For any unit vector $\varphi \in (\mathbb{C}^d)^{\otimes m}$,

$$\left\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\right\|_{(\ell_2^{d^2})^{\otimes_\pi m}} \geq \frac{1}{\|\varphi\|_{(\ell_2^d)^{\otimes_\epsilon m}}}. \tag{6}$$

- If in addition φ is non-negative (meaning that its coefficients in the canonical basis of $(\mathbb{C}^d)^{\otimes m}$ are all non-negative), then

$$\left\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\right\|_{(\ell_2^{d^2})^{\otimes_\pi m}} \geq \frac{1}{\|\varphi\|^2_{(\ell_2^d)^{\otimes_\epsilon m}}}. \tag{7}$$

W-state :$|w\rangle := \frac{1}{\sqrt{3}}(|112\rangle + |121\rangle + |211\rangle) \in (\mathbb{C}^2)^{\otimes 3}$.

- entangled and $\||w\rangle\langle w|\|_{(S_1^2)^{\otimes_\pi 3}} = \|w\|^2_{(\ell_2^2)^{\otimes_\pi 3}} = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$.
- $\left\|\mathcal{R}^{\otimes 3}(|w\rangle\langle w|)\right\|_{(\ell_2^4)^{\otimes_\pi 3}} \geq \frac{1}{\|w\|^2_{(\ell_2^2)^{\otimes_\epsilon 3}}} = \frac{1}{(2/3)^2} = \frac{9}{4}$
- so, $\left\|\mathcal{R}^{\otimes 3}(|w\rangle\langle w|)\right\|_{(\ell_2^4)^{\otimes_\pi 3}} = \||w\rangle\langle w|\|_{(S_1^2)^{\otimes_\pi 3}} = \frac{9}{4} > 1$.

Observation: the same holds for multipartite pure states having a generalized Schmidt decomposition $|\varphi\rangle = \sum_{k=1}^{r} \sqrt{\lambda_k} \left|e_k^1 \cdots e_k^m\right\rangle$.

$$\left\|\mathcal{R}^{\otimes m}(|\varphi\rangle\langle\varphi|)\right\|_{(\ell_2^{d^2})^{\otimes_\pi m}} = \||\varphi\rangle\langle\varphi|\|_{(S_1^d)^{\otimes_\pi m}}.$$

Conclusions

Introducing a new paradigm for entanglement detection in bipartite/multipartite quantum system based on entanglement testers:

- Entanglement testers: contractions $\mathcal{E} : S_1^d \to \ell_2^n$
- reduce entanglement problem of mixed q. states to that of pure q. s.
- **Entanglement criterion based on projective tensor norm**: if the $\mathcal{E}_i$'s are testers, then $\|\mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_m(\rho)\|_{\ell_2^{n_1} \otimes_\pi \cdots \otimes_\pi \ell_2^{n_m}} > 1 \implies \rho$ is entangled.
- extends to multipartite case criteria (RLN, ESIC POVM, SSC)
- reformulate the theory for other criteria: reduction, entanglement criteria based on correlation matrix.
- prove the conjecture that $\mathcal{R} \subset \mathcal{S}$;
- completeness for mixed bipartite states and pure multipartite states: to do: study the case of multipartite mixed states and enhanced entanglement criteria

Thank you!