Injective norm of real and complex random tensors

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Joint work with Stephane Dartois

I) Motivations

Goal: "How entangled is a random quantum state?"

i.e. how close is a random quantum state $|\psi\rangle$ (element of $(\mathbb{C}^d)^{\otimes p}$)

to being separable = rank-one, $|\psi\rangle \overset{?}{\approx} |\psi_1, \dots, \psi_p\rangle := |\psi_1\rangle \otimes \cdots \otimes |\psi_p\rangle$?

Main thm: Upper bound for one formalization of this Q.

Interesting because:

1. Method is different from previous bounds (comes from spinglasses)
2. Leading-order constant is conjecturally tight (Guillaume's α_p)

How to formalize:

① Step 1: "Quantum state" $\rightsquigarrow$ "Tensor"

Interpret quantum states as tensors $T \in (\mathbb{C}^d)^{\otimes p}$,

$$T_{i_1 \dots i_p} = \langle \psi | e_{i_1}, \dots, e_{i_p} \rangle \in \mathbb{C} \quad \text{for } i_j \in [\![1, d]\!]$$

↑ standard basis vectors

$\rightsquigarrow$ "How far is a random tensor from being separable?"

② Step 2: "far from separable" $\rightsquigarrow$ "injective norm"

Matrix case: Sep. = rank one, $T = xy^T$, so one geometric notion of "close to sep." is "large overlap with rank-one matrices"

$$\max_{\|x\|=\|y\|=1} |\langle T, xy^T \rangle_{frob}| = \max_{\|x\|=\|y\|=1} |\langle y, Tx \rangle| = \|T\|_{op}$$

$$= \max_{\|x\|=\|y\|=1} \Big| \sum_{ij} T_{ij} x_i y_j \Big|.$$

Tensor analogue: "injective norm"

$$\|T\|_{inj} = \max_{\|x^{(1)}\|=\cdots=\|x^{(p)}\|=1} \Big| \sum_{i_1 \cdots i_p = 1}^{d} T_{i_1 \cdots i_p} x^{(1)}_{i_1} \cdots x^{(p)}_{i_p} \Big|$$

Geometric entanglement of T $\leftrightsquigarrow$ $-\log \|T\|_{inj}$.

$\rightsquigarrow$ "What is the injective norm of a random tensor T?"

③ Step 3: "random" $\rightsquigarrow$ "Gaussian"

- Many prob. measures on tensors
- Simplest: $T_{i_1 \cdots i_p}$ IID standard Gaussians

(quantum: often interested in normalized version

$$\tilde{T}_{i_1 \cdots i_p} = \frac{T_{i_1 \cdots i_p}}{\gamma}, \qquad \gamma^2 = \sum_{i_1 \cdots i_p = 1}^{d} |T_{i_1 \cdots i_p}|^2$$)

almost deterministic for our purposes, so focus on unnormalized T

$\rightsquigarrow$ "What is the injective norm of a Gaussian tensor T?"

Rmks:

① $$\|T\|_{inj} = \max_{\|x^{(i)}\|=1} \left| \sum_{i_1,\dots,i_p=1}^{d} T_{i_1\cdots i_p} x^{(1)}_{i_1} \cdots x^{(p)}_{i_p} \right|$$

$$=: \max_{\|x^{(i)}\|=1} \left| f_T(x^{(1)},\dots,x^{(p)}) \right|$$

So is the maximum of a nice Gaussian process (random polynomial!) on a product of spheres.

② Diagonal $f_T(x,\dots,x)$ is a pure spherical p-spin glass (if $T_{i_1\cdots i_p}$ real)

Many choices from here:

① Entries $T_{i_1\cdots i_p}$ real or complex

easier — quantum motivation

② Symmetry:

- T nonsymmetric (genuinely IID)
- T symmetric ($T_{\pi(i_1)\cdots\pi(i_p)} = T_{i_1\cdots i_p}$)

③ Scaling limits

- p fixed, $d\to\infty$ (spin glasses)
- d fixed, $p\to\infty$ (quantum information)

II History

- $p=2$ is the operator norm of a Gaussian matrix (classical)
- Symmetric case benefits from

<u>Thm:</u> (Kellogg 1928, van der Corput-Schaake 1935): Deterministically,

$$\|T\|_{inj} = \max_{\|x^{(i)}\|=1} \left|\langle T, x^{(1)} \otimes \cdots \otimes x^{(p)} \rangle\right| = \max_{\|x\|=1} \left|\langle T, x^{\otimes} \cdots \otimes x \rangle\right|$$

$\Rightarrow$ Real, symmetric, p fixed $d \to \infty$ case is ground state of spherical p-spin

$\Rightarrow$ completely solved (Crisanti - Sommers 1995, Auffinger - Ben Arous - Černý 2013, Subag 2017)

- Otherwise, ε-net techniques give correct order but not correct constant:
  - Gross - Flammia - Eisert 2009: When $d=2$, $\mathbb{C}$, normalized, $\exists c, C$

$$\mathbb{P}\left(c\frac{\sqrt{\log p}}{2^{p/2}} \le \|\tilde{T}\|_{inj} \le C\frac{\sqrt{\log p}}{2^{p/2}} \right) \xrightarrow{p\to\infty} 1.$$

(Our result: $C = 1+\varepsilon$)

  - Tomioka - Suzuki 2014, Nguyen - Drineas - Tran 2015: $\mathbb{R}$, unnormalized, finite-d,p bounds roughly like

$$\|T\|_{inj} \le C\sqrt{dp\log p}$$

(Our result: $C = 1+\varepsilon$)

- <u>Numerics:</u> Fitter - Lancien - Nechita 2022: Both $\mathbb{R} + \mathbb{C}$, both sym./non, $p=3$ or 4 and $d \to \infty$.

III Results

<u>Thm:</u> (Dartois–M. 2024): High-probability upper bound for

- nonsymmetric
- $p \to \infty$ or $d \to \infty$
- $\mathbb{R}$ or $\mathbb{C}$
- normalized or unnormalized

Leading constant is explicit, conjecturally tight, and in special cases agrees with

- Numerics of Fitter–Lancien–Nechita
- Simultaneous physics work of Sasakura 2024
- Conditional spin-glass results of Subag.

One example of the bound: $\mathbb{R}$, unnorm., p fixed, $d \to \infty$:

- Previously: $\quad \| T_{sym} \|_{inj} \approx \sqrt{d}\, E_0(p)$,

  where $E_0(p)$ is implicitly defined, $\dfrac{E_0(p)}{\sqrt{\log p}} \to 1$ as $p \to \infty$

- Us: $\quad \| T \|_{inj} \leq \sqrt{dp}\, E_0(p)(1+\varepsilon)$.

<u>Rmks:</u>

1. Same $E_0(p)$! Simple explanation?
2. Verifies folklore notion "nonsymmetric states are more entangled than symmetric states"

IV Methods (unnormalized real case)

- $M = (S^{d-1})^{\otimes p}$ is a nice manifold, and

$$\sup_{x \in M} f_T(x) = \inf \{ u : f \text{ has no critical points above } u \}$$

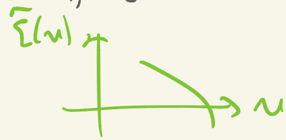
So if we can compute

$$\tilde{\Sigma}(u) = \log \mathbb{E}[\mathrm{Crt}(f_T, u)] = \log \mathbb{E}\left[\# \{ \vec{x} : \nabla f_T(\vec{x}) = 0 \text{ and } f_T(\vec{x}) \geq u \}\right]$$

then

$$\mathbb{P}(\|T\|_{inj} \geq u) = \mathbb{P}(\mathrm{Crt}(f_T, u) \geq 1) \leq \mathbb{E}[\mathrm{Crt}(f_T, u)]$$

$$\rightsquigarrow \|T\|_{inj} \leq \text{zero of } \tilde{\Sigma}(u)$$



How to compute $\tilde{\Sigma}(u)$?

- Kac-Rice formula:

$$\mathbb{E}[\mathrm{Crt}(f_T, u)] = \int_M d\sigma \, \mathbb{E}\Big[\overbrace{|\det(\nabla^2 f_T(\sigma))}^{\text{Hessian at a critical point}} \mathbb{1}\{ f_T(\sigma) \geq u \} \Big| \nabla f_T(\sigma) = 0 \Big]$$

$$\times \underbrace{\varphi_\sigma(0)}_{\text{"}= \mathbb{P}(\nabla f_T(\sigma) = 0)\text{"}} d\sigma$$

$\Rightarrow$ Turns into "How does the determinant of a random matrix behave?"

- Standard spin-glass idea, but the random matrices are harder here

e.g. for real case $p=3$,

$$B = \begin{pmatrix} \boxed{0} & & N(0,1) \\ & \boxed{0} & \\ N(0,1) & & \boxed{0} \end{pmatrix}$$

zero blocks are technically difficult

build on Ben Arous-Bourgade-M. 2021

Complex: Worse

High-level reason why $E_0(p)$ still appears:

If matrix W has $W_{ij} \sim \mathcal{N}(0, \sigma_{ij}^2)$, histogram is still semicircular

if all row sums are equal: $\sum_j \sigma_{ij}^2$ does not depend on i.

V Summary

- Quantum information: Kac-Rice can give finer constants on injective norms than ε-nets, and recent RMT advances help
- Spin glasses: Quantum motivation for ground state of unusual spin glasses:
    - Multispecies
    - p-spin for $p \to \infty$
    - Complex

Open:

1. Matching lower bound? (stay tuned)
2. Non-Gaussian models of randomness, like MPS (not clear that Kac-Rice applies)
3. Add a signal? (Non-symmetric analogue of tensor PCA results? This is the pure noise case.)