Laboratoire de Physique Théorique et Modèles Statistiques

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IHP, October 2024

Random Tensors & Related topics

# ENERGY LANDSCAPES FROM RANDOM TENSORS: CORRELATIONS BETWEEN TRIPLETS OF STATIONARY POINTS

Based on work with A. Pacco, A. Rosso

cnrs

université PARIS-SACLAY GRADUATE SCHOOL Physique

PhOM Physique des Ondes et de la Matière

# THIS TALK: STAT-PHYS. DICTIONARY

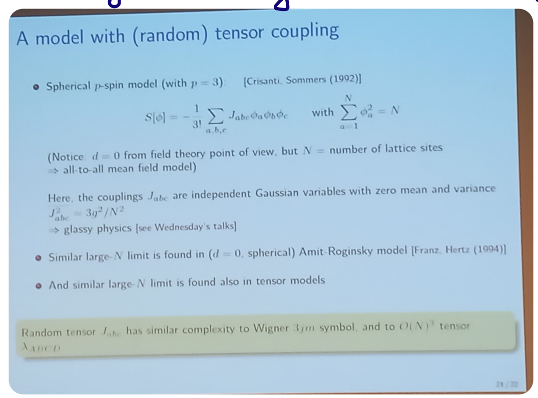
- High-D random functions from random tensors, $\mathcal{E}(s)$

  $\Longrightarrow$ energy landscapes, $s$ = configuration

  $s \in \Omega_N \subset \mathbb{R}^N$, $N \gg 1$

**same as Leticia's talk**

Monday's talk by D. Benedetti




- Study distribution of stationary points in configuration space

  $\Longrightarrow$ landscape's geometry

- Motivation: dynamics beyond $N \to \infty$ limit

  $\Longrightarrow$ optimization at very large timescales, beyond "mean-field"

**relaxational (not Hamiltonian) dynamics**

**same regime as "single mode analysis" of Leticia's talk**

# FROM TENSORS TO HIGH-D LANDSCAPES

High-dimensional ($N \gg 1$) random functions obtained contracting tensors with vectors:

$$\mathcal{E}(s) = \sqrt{\frac{p!}{2N^{p-1}}} \sum_{i_1 \le \dots \le i_p} a_{i_1 \dots i_p} s_{i_1} \cdots s_{i_p}$$

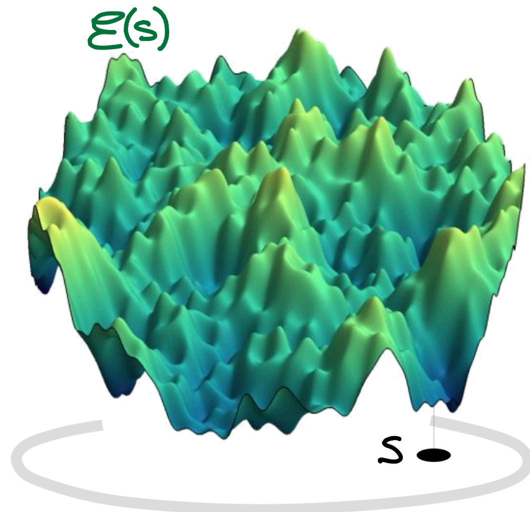
$$s = (s_1, \dots, s_N) \in \mathcal{S}_{N-1} = \left\{ s \in \mathbb{R}^N : \sum_{i=1}^{N} s_i^2 = N \right\}$$

$a_{i_1 \dots i_p}$ symmetric, random Gaussian tensor

$\langle a_{i_1 \dots i_p} \rangle = 0$, $\langle a^2_{i_1 \dots i_p} \rangle = 1$, $p \geq 3$

I will call: $s$ the configuration, $\mathcal{S}_{N-1}$ the configuration space, $a_{i_1 \dots i_p}$ the randomness, $\mathcal{E}(s)$ the energy landscape.

These functions are typically ($\mathbb{P} \xrightarrow{N \to \infty} 1$) HIGHLY NON-CONVEX: plenty of local minima & saddles of any index ($\nabla \mathcal{E}(s) = 0$)




configuration space dimension $N \gg 1$

"Plenty" = $\mathcal{N} \sim O(e^N)$    $N \gg 1$

number stationary points

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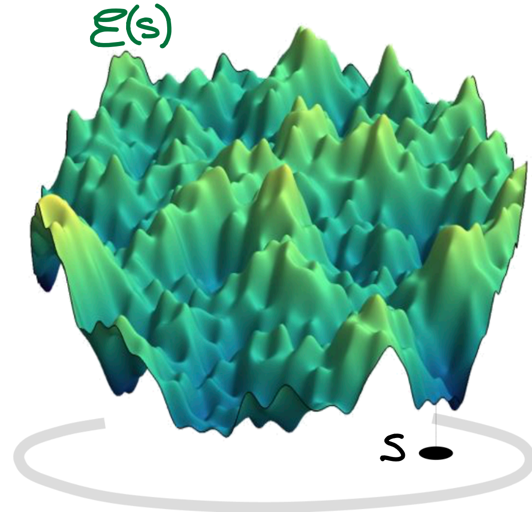



configuration space dimension $N \gg 1$

- In stat phys: good models of energy landscapes of complex systems, e.g. glasses

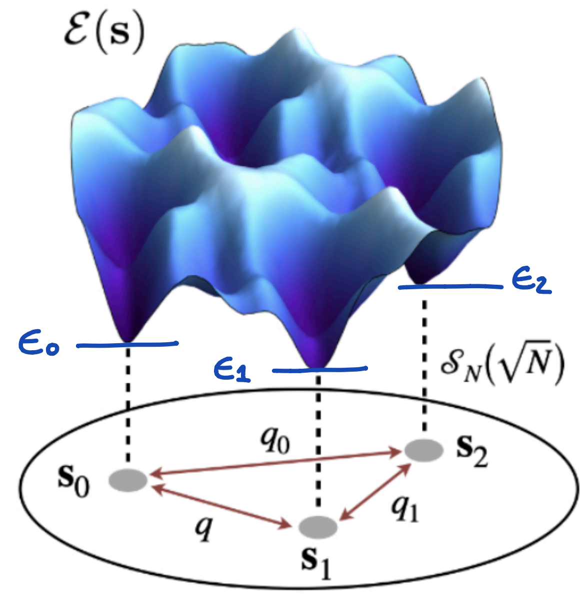
  Local minima = metastable states

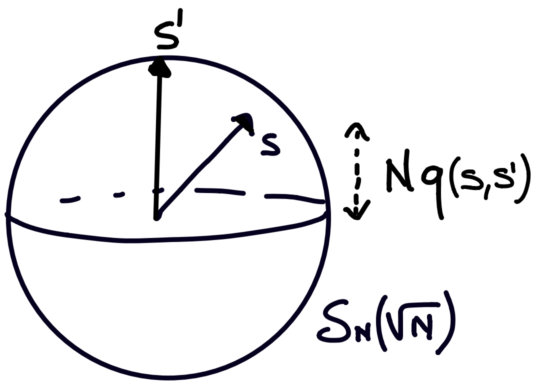
- In high-D inference: tensor PCA via maximum likelihood.

# WHAT WE DO: A CONDITIONAL COUNTING

Joint (conditional) distribution of triplets of stationary points (minima, saddles) as a function of:

- energy: $\epsilon_a = \mathcal{E}(\mathbf{s}_a)/N \qquad a = 0,1,2$
- position: $q(\mathbf{s}_a, \mathbf{s}_b) = (\mathbf{s}_a \cdot \mathbf{s}_b)/N$ "overlaps"




More precisely, given the random variable:

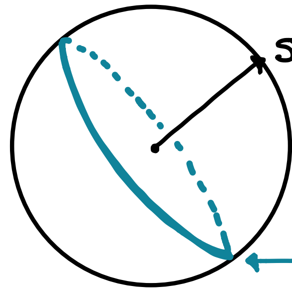
$\mathcal{N}_{\mathbf{s}_0 \mathbf{s}_1}(\epsilon_2, q_1, q_0)$ = number stat. points $\mathbf{s}_2$ $(\nabla \mathcal{E}(\mathbf{s}_2) = 0)$ at energy density $\epsilon_2$, at fixed overlap from two other stat. points $\mathbf{s}_0, \mathbf{s}_1$ at energies $\epsilon_0, \epsilon_1$ and $\mathbf{s}_0 \cdot \mathbf{s}_1 = Nq$.

we compute

$$\boxed{\Sigma^{(3)}(\epsilon_2, q_1, q_0 \,|\, \epsilon_1, \epsilon_0, q) = \lim_{N \to \infty} \frac{1}{N} \left\langle \log \mathcal{N}_{\mathbf{s}_0 \mathbf{s}_1}(\epsilon_2, q_1, q_0) \right\rangle_{\mathbf{s}_0, \mathbf{s}_1}}$$

# IN HIGH-D, LOCALITY HAS TO BE ENFORCED

In high-D "entropy pushes you far-away": random pairs of stationary points are typically orthogonal

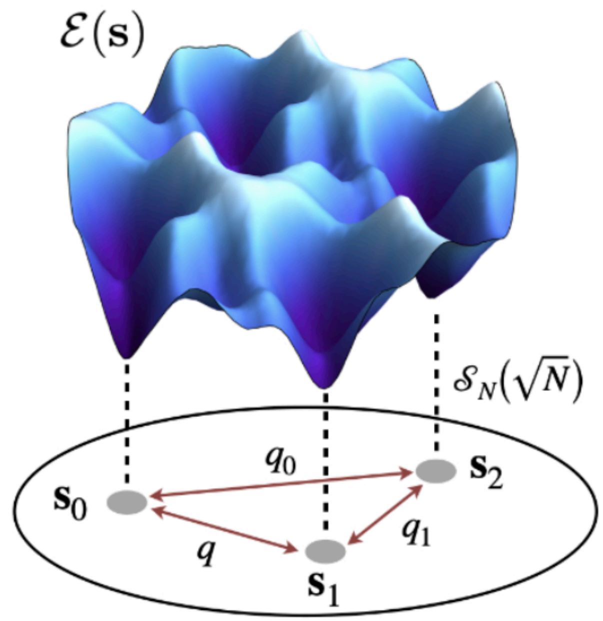



- One-point complexity: Unbiased counting describes "equator."

$$\Sigma(\epsilon) = \lim_{N\to\infty} \langle \log \mathcal{N}(\epsilon) \rangle$$

- Two-point complexity: distribution of stationary points $s_1$ "close" to $s_0$. LOCAL GEOMETRY

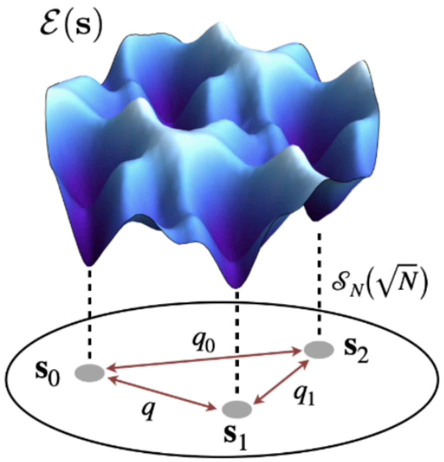
$$\Sigma^{(2)}(\epsilon_1 | q, \epsilon_0) = \lim_{N\to\infty} \frac{1}{N} \langle \log \mathcal{N}_{s_0}(\epsilon_1, q) \rangle_{s_0}$$




- Three-point complexity: LOCAL CORRELATIONS

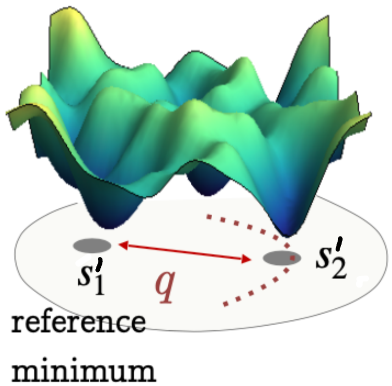
$$\Sigma^{(3)}(\epsilon_2, q_1, q_0 | \epsilon_1, \epsilon_0, q) = \lim_{N\to\infty} \frac{1}{N} \langle \log \mathcal{N}_{s_0 s_1}(\epsilon_2, q_1, q_0) \rangle_{s_0, s_1}$$

# READING LOCAL CORRELATIONS FROM $\Sigma^{(3)}$




$\Sigma^{(3)}(\epsilon_2, q_1, q_0 | \epsilon_1, \epsilon_0, q) \Longrightarrow$ landscape structure (distribution of $s_2$) around stationary point $s_1$ of energy $\epsilon_1$, in the vicinity of $s_0$

To be compared with:

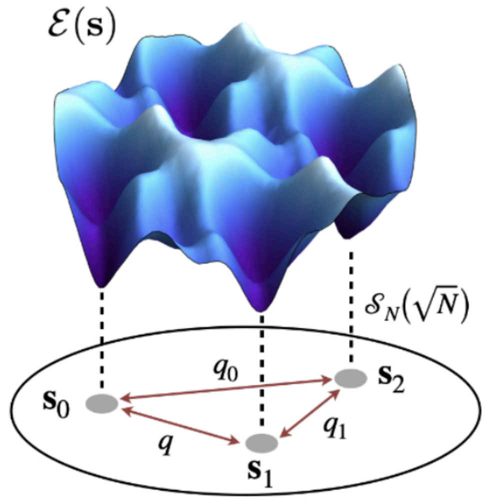



$\Sigma^{(2)}(\epsilon_2, q_1 | \epsilon_1) \Rightarrow$ landscape structure around stationary point $s_1'$ of energy $\epsilon_1$, not conditioned to be close to $s_0$.

How much $s_0$ deforms surrounding landscape ?

# LOCALITY HAS TO BE ENFORCED APPROPRIATELY

The configurations $s_0, s_1, s_2$ are not on the same footing: $s_0$ choosen with no constraint, $s_1$ constrained by $s_0$, $s_2$ constrained by both.




- Encoded in way one averages over $s_0, s_1$:

$$\Sigma^{(3)}(\epsilon_2, q_1, q_0 \,|\, \epsilon_1, \epsilon_0, q) = \lim_{N \to \infty} \frac{1}{N} \langle \log \mathcal{N}_{s_0 s_1}(\epsilon_2, q_1, q_0) \rangle_{s_0, s_1}$$

average over $s_0, s_1$ with appropriate measure

- Requires REPLICA METHOD. In jargon, "quenched calculation".

  Similar to: FRANZ, PARISI 1998

- Combined with Kac-Rice formalism (random matrix theory)

  short review: VR, FYODOROV 2023

## PART I: RANDOM LANDSCAPES & GLASSYNESS

On the statistical-physics history of the model

An open problem: dynamics beyond $N \to \infty$

## PART II: FROM DYNAMICS TO GEOMETRY, AND BACK

Jumps between minima: memory & correlations?

Three-point complexity: landscape's transitions

Back to dynamics

Kac-Rice formalism & random matrices

RANDOM LANDSCAPES & GLASSYNESS

# On the statistical-physics history of the model

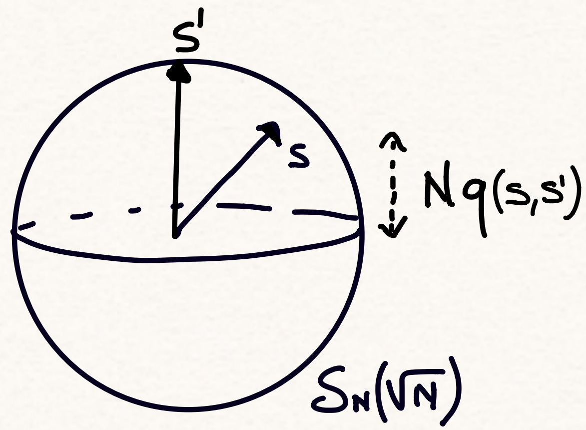
# THE SPHERICAL P-SPIN : THE "BABY GLASS"

Gaussian random landscape in high-D

$$\mathcal{E}[s] = \sqrt{\frac{p!}{2N^{p-1}}} \sum_{i_1 < \dots < i_p} a_{i_1 \dots i_p} S_{i_1} \cdots S_{i_p} \qquad (p \geq 3)$$

$$S = (S_1, \dots, S_N) : \sum_{i=1}^{N} S_i^2 = N$$

$$a_{i_1 \dots i_p} = \text{Gaussian iid} \sim \mathcal{N}(0,1)$$



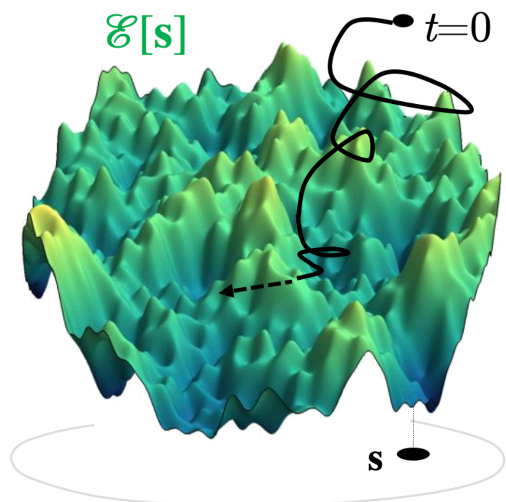

Isotropic statistics

$$\langle \mathcal{E}[s] \rangle = 0 \qquad \langle \mathcal{E}[s]\, \mathcal{E}[s'] \rangle = \frac{N}{2} \left[ q(s,s') \right]^p$$

$$q(s,s') = \frac{s \cdot s'}{N} = \text{"OVERLAP"} - \text{similarity}$$

GROSS, MEZARD 1984

CRISANTI, SOMMERS 1995

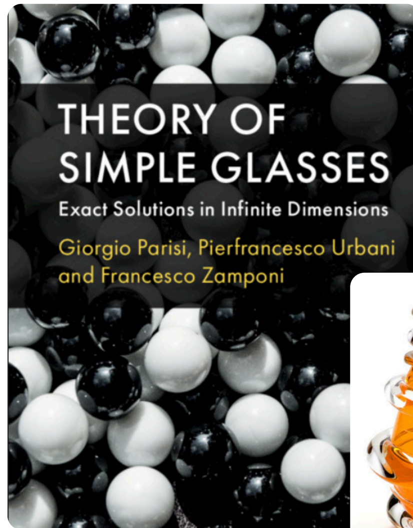



configuration space of dimension $N \gg 1$

# A MEAN-FIELD MODEL OF GLASSES

Spherical p-spin captures distinctive features of glasses
KIRKPATRICK, THIRUMALAI, WOLYNES 1989

$D \to \infty$ theory of glasses solved in 2014-2020 $\Longrightarrow$
PARISI, URBANI, ZAMPONI, KURCHAN, CHARBONNEAU 2017

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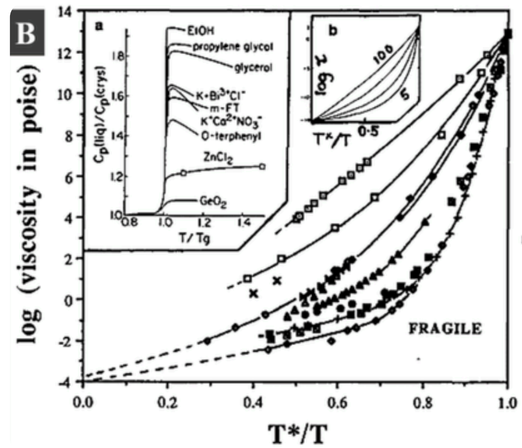
Which "distinctive features"?

- ▨ A rich energy landscape, with plenty of local minima, many of which (i) almost equivalent energy, (ii) quite different configurations ($q \neq 1$)

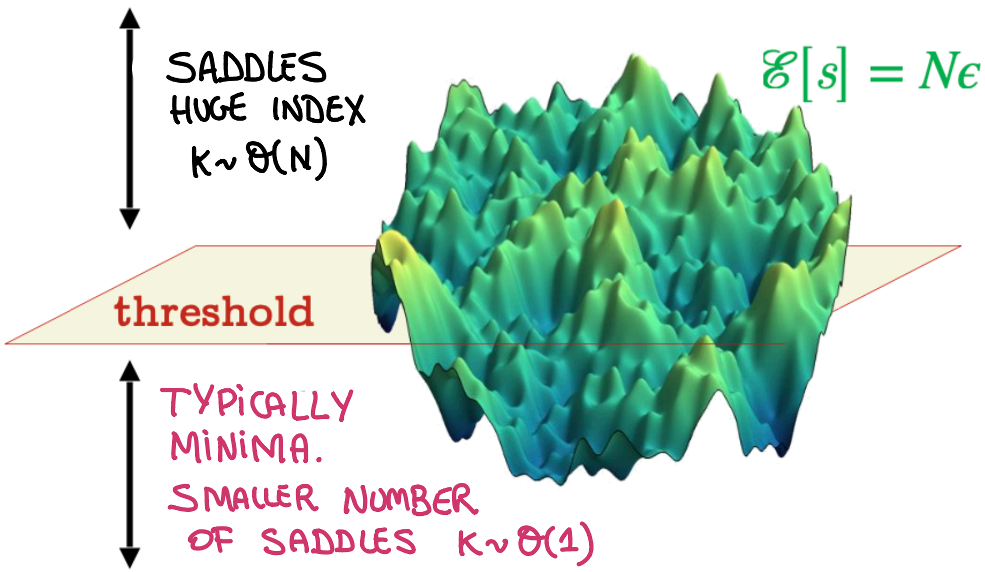
- ▨ A solvable dynamics when $N \to \infty$, showing slowdown and absence of equilibration.




ANGELL 1995

# P-SPIN, THE KNOWN: THE LANDSCAPE

$\mathcal{N}(\epsilon) = e^{N\Sigma(\epsilon)+o(N)}$ number of stationary points $\nabla\mathcal{E}(s)=0$, $\mathcal{E}(s)=N\epsilon$. Random.
Complexity $\Sigma(\epsilon) = \lim_{N\to\infty} \frac{\log \mathcal{N}(\epsilon)}{N}$ is self-averaging. Known explicitly!




- Critical energy $\epsilon_{th}$ marks transition between minima ($\epsilon \leq \epsilon_{th}$) and saddles: stability transition
- At $\epsilon_{th}$, marginal stability
- Below threshold: local minima separated by extensive barriers.

CAVAGNA, GIARDINA, PARISI 1997, 1998
AUFFINGER, BEN AROUS, CERNY 2013
SUBAG 2015

"Typically" = most stationary points.
Exponential majority.

# P-SPIN SOLUTION (II): THE DYNAMICS

- Langevin dynamics: $\frac{ds(t)}{dt} = -\nabla \mathcal{E}(s(t)) + \sqrt{2T}\,\eta(t)$ ⟶ NOISE-EXPLORATION

  GRADIENT DESCENT-OPTIMIZATION

- Can derive exact dynamical equations when $N \to \infty$

  Closed equations for two-point functions: DYNAMICAL MEAN-FIELD THEORY (DMFT)

  $$\{s_i(t)\}_{i=1}^{N} \longrightarrow C(t,t') = \frac{1}{N}\sum_{i=1}^{N} s_i(t)\, s_i(t'),\quad \epsilon(t) = \frac{1}{N}\mathcal{E}(s(t)), \dots$$

  SOMPOLINSKY, ZIPPELIUS 1981

  CRISANTI, HORNER, SOMMERS 1993

  **—> as in Leticia's talk for Hamiltonian dynamics**

- Scheme now used in variety of high-D problems in computer science, machine learning, inference Review: CUGLIANDOLO 2023

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- Scheme now used in variety of high-D problems in computer science, machine learning, inference   Review: CUGLIANDOLO 2023

Asymptotic form of solution for $C(t,t')$, $R(t,t')$ found in
CUGLIANDOLO, KURCHAN 1993   Notice: first $N \to \infty$, then $t, t' \longrightarrow \infty$

# THE P-SPIN PARADIGM for GLASSY DYNAMICS

DMFT solutions give **A LOT OF INSIGHT** on relaxational, out-of-equilibrium dynamics: separation of timescales, aging, effective temperatures, fluctuation-dissipation violation, weak ergodicity-breaking scenario, "quasi-equilibrium" dynamics

CUGLIANDOLO, KURCHAN 1993 FRANZ, PARISI 2013
REVIEW: BOUCHAUD, CUGLIANDOLO, KURCHAN, MEZARD 1998

## Back to particles:

DMFT equations have **SAME STRUCTURE AS MODE COUPLING EQUATIONS** for supercooled liquids: (i) perturbative, diagrammatic expansion of Langevin (ii) neglect all vertex corrections, keep only line corrections. Exact for spherical p-spin. KIRKPATRICK, THIRUMALAI, WOLYNES 1989

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## Back to particles: $\Rightarrow$ like melonic diagrams

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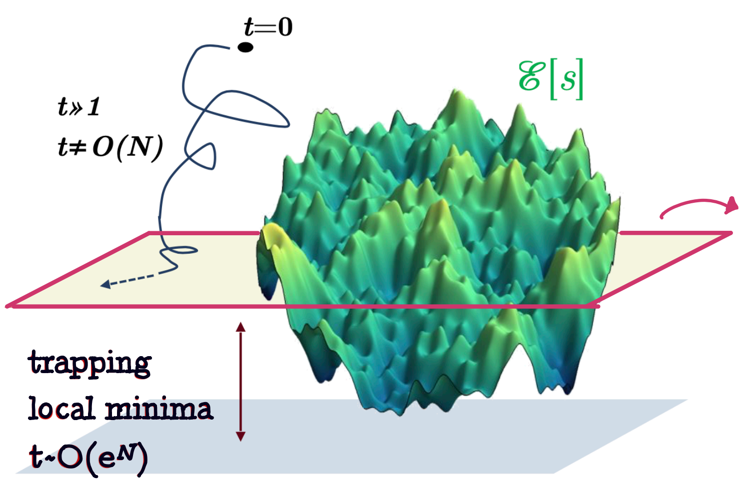
# MEAN-FIELD DYNAMICS & METASTABILITY

Dynamics from random $s(t=0)$, weak noise ($T \ll 1$)

DMFT solution describes out-of-equilibrium relaxation towards threshold.
Slow dynamics, out-of-equilibrium, aging.



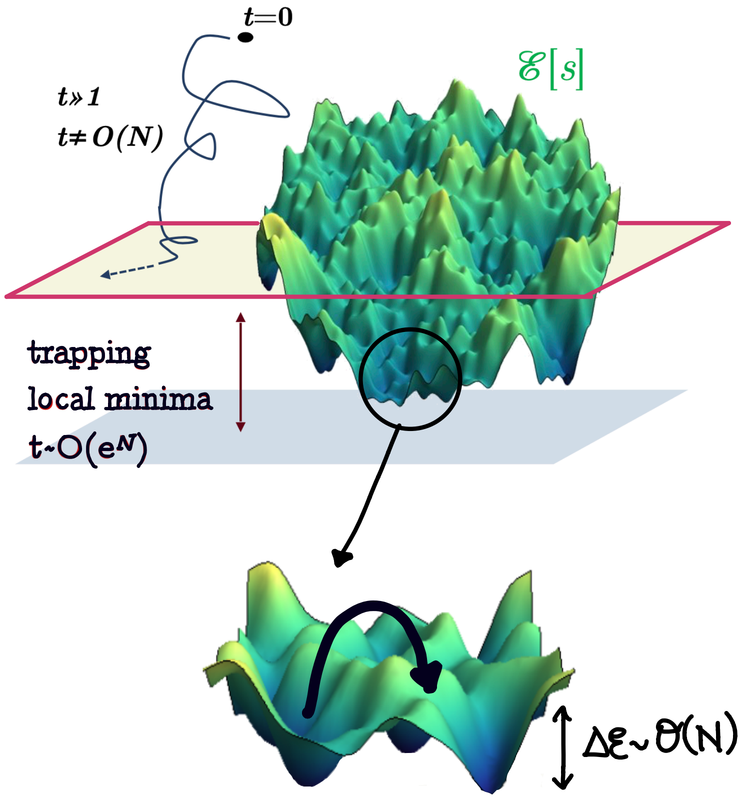

THRESHOLD. DMFT dynamics stuck here!
where first local minima appear.
LANDSCAPE-DYNAMICS CONNECTION

Asymptotic convergence to threshold also math-proven: SELLKE 2023

## RANDOM LANDSCAPES & GLASSYNESS

An open problem: dynamics beyond $N \to \infty$ (= mean field)

# P-SPIN, THE UNKNOWN: ACTIVATED DYNAMICS




- Mean-field, $N \to \infty$. local minima below $\epsilon_{th}$ separated by extensive barriers $\Delta\epsilon \sim \Theta(N)$: true trapping states when $N \to \infty$! Reach only highest ones.

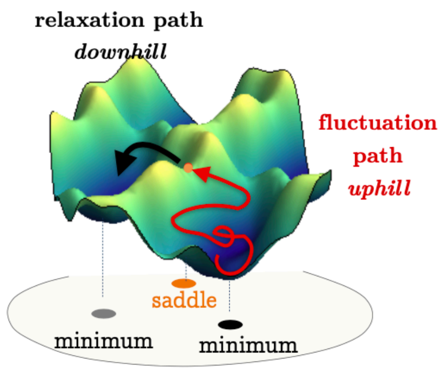
- $N \gg 1$, but finite: escape processes possible, at timescales $\tau \sim e^{\Delta\epsilon} \sim e^{N}$. Landscape explored via ACTIVATED DYNAMICS. Dynamical transition becomes CROSSOVER.

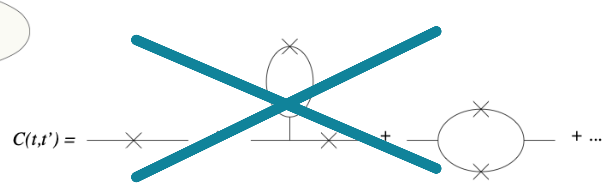
Activated jumps not captured by DMFT ($t \to \infty$ after $N \to \infty$. Here, need $t \sim e^N$)

Question: Theory of activated dynamics in high-D

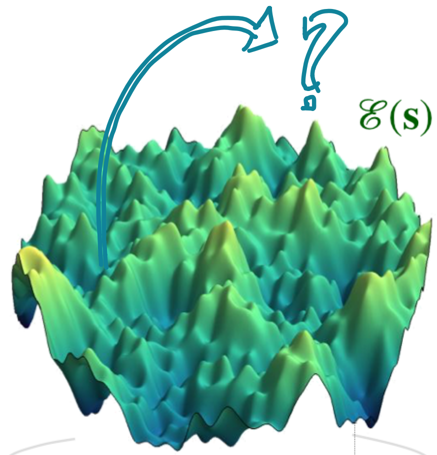
# ACTIVATED DYNAMICS IN HIGH-D: WHY DIFFICULT

No 1/N expansion: activated processes are non-perturbative
Instantons of dynamical theory




High-D: entropy!
Exponentially-many minima, saddles, ....
which path(s) chosen by the system?




Approaches:

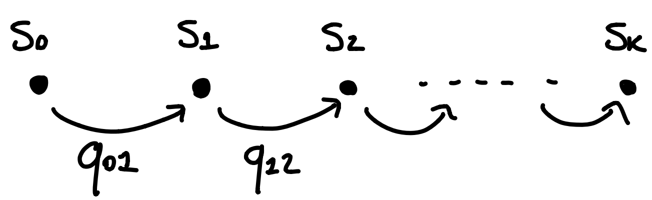
- large deviations of DMFT, instantons — RIZZO 2020, LOPATIN, IOFFE 1999, VR, BIROLI, CAMMAROTA 2021
- effective "jump processes" in configuration space → rest of talk

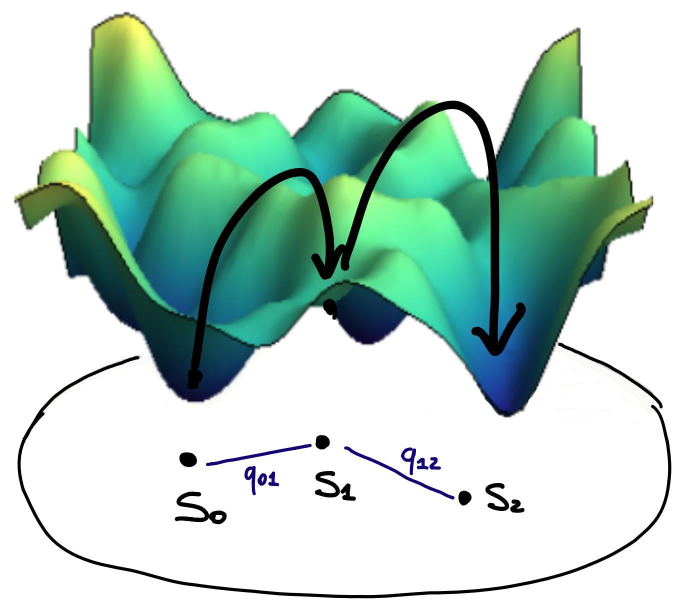
# Jumps between minima: memory & correlations?

# A RANDOM JUMP PROCESS AMONG MINIMA

- Sequence of minima $s_0, s_1, s_2 \dots s_K$ with energy densities $\epsilon_0, \epsilon_1, \epsilon_2 \dots$

- assume that transition rates depend on energies, and on distances in configuration space: $q_{ab} = \frac{1}{N} s_a \cdot s_b$
(isotropy)




$$\mathcal{T}_{s_a}\left(\{\epsilon_a\}_{a=1}^{K}, \{q_{ab}\}_{a\leq b}\right) = t^{(1)}_{s_0,s_1}(\varepsilon_1 | \varepsilon_0; q_{01})\, t^{(2)}_{s_0,s_1,s_2}(\varepsilon_2, q_{21} | \varepsilon_1, \varepsilon_0; q_{10}, q_{20}) \cdots t^{(K)}_{s_a}(\varepsilon_K, q_{K,K-1} | \varepsilon_{K-1}, \varepsilon_{K-2}, \dots; q_{K-1,K-2} \cdots)$$

$$\mathcal{G}_{s_a}\left(\{\epsilon_a\}_{a=1}^{K}, \{q_{ab}\}_{a\leq b}\right) = t^{(1)}_{s_0,s_1}(\varepsilon_1|\varepsilon_0; q_{01})\, t^{(2)}_{s_0,s_1,s_2}(\varepsilon_2, q_{21}|\varepsilon_1,\varepsilon_0; q_{10}, q_{20}) \cdots t^{(K)}_{s_a}(\varepsilon_K, q_{K,K-1}|\varepsilon_{K-1},\varepsilon_{K-2},\dots; q_{K-1,K-2}\cdots)$$

Question: how memory (correlations) propagates along the chain? Correlations between two consecutive jumps?

Any case in which $t^{(2)}_{s_0,s_1,s_2}(\varepsilon_2, q_{21}|\varepsilon_1,\varepsilon_0; q_{10}, q_{20}) \to t^{(2)}_{s_1,s_2}(\varepsilon_2, q_{12}|\varepsilon_1)$ : "MEMORYLESS" ?

Notice: memoryless dynamics assumed in effective models of activated dynamics in complex landscapes

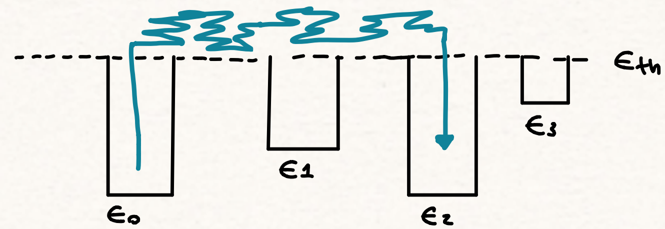
"TRAP MODELS" DYRE '87, BOUCHAUD '92, GAYRARD 2019

# SIMPLEST SCENARIO: TRAP MODELS & REM

Trap model: memoryless dynamics with

$$t^{(n)}(\epsilon_n, q_{n,n-1} | \{\epsilon, q\}) \longrightarrow t(\epsilon_{n-1}) \propto e^{-\beta(\epsilon_{th} - \epsilon_{n-1})}$$

$\epsilon_{th}$ fixed



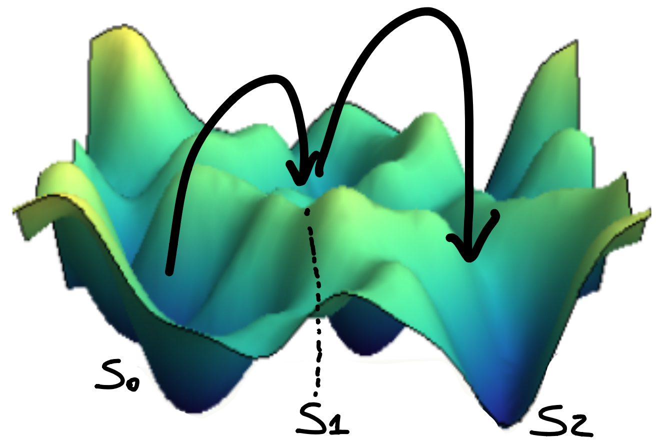

- Solved explicitly: effective temperatures, aging... DYRE '87, BOUCHAUD '92
- Gives correct description for RANDOM ENERGY MODEL (no correlations in landscape)
  GAYRARD 2019, BAITY-JESI et al 2018, CERNY and WASSMER 2015
- Is activated dynamics "trap-like" for energy landscapes with local correlations? Not straightforward.
  STARIOLO, CUGLIANDOLO 2019, 2020

# FROM DYNAMICS TO GEOMETRY

Two jumps: $\tau = t^{(1)}_{s_0,s_1}(\varepsilon_1|\varepsilon_0;q_{01})\, t^{(2)}_{s_0,s_1,s_2}(\varepsilon_2,q_{21}|\varepsilon_1,\varepsilon_0;q_{10},q_{20})$

- For fixed energies $\varepsilon<\varepsilon_{th}$, many available minima. Consider jumps to CLOSEST minima at those energies.

  WHY? Energy barrier decreasing in overlap $q$

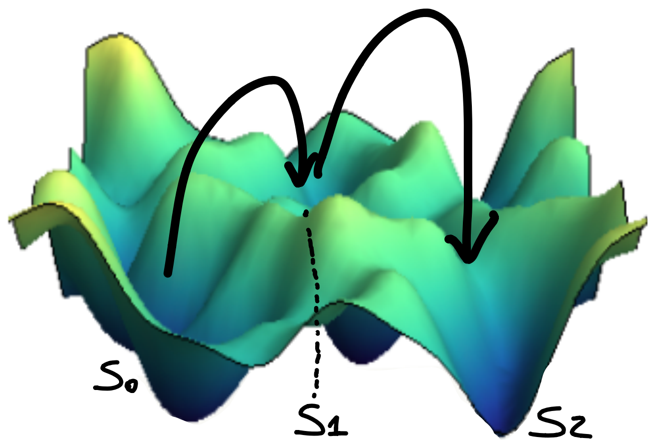



- Rates depend on energies only:

$$q_{01} \to q^{(1)}_{MAX}(\epsilon_1|\epsilon_0) \qquad\qquad t^{(1)}(\epsilon_1|\epsilon_0,q_{01}) \to t^{(1)}(\epsilon_1|\epsilon_0)$$

$$q_{12} \to q^{(2)}_{MAX}(\epsilon_2|\epsilon_1,\epsilon_0) \quad \Longrightarrow \quad t^{(2)}(\epsilon_2,q_{21}|\epsilon_1,\epsilon_0,q_{10},q_{20}) \to t^{(2)}(\epsilon_2|\epsilon_1,\epsilon_0)$$

# FROM DYNAMICS TO GEOMETRY



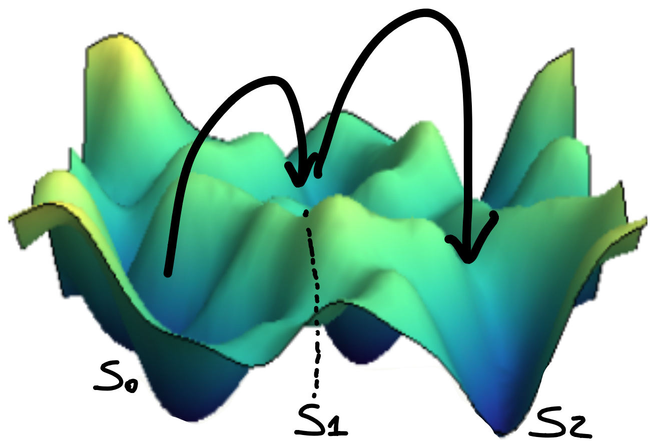

- **SCENARIO 1: "AVALANCHES".** Big activated jump followed by sequence of smaller rearrangements. Correlations.

  $$t^{(2)}(\varepsilon_2|\varepsilon_1,\varepsilon_0) > t^{(1)}(\varepsilon_2|\varepsilon_1)$$

- **SCENARIO 2: "MEMORYLESS".** Second jump uncorrelated to the first.

  $$t^{(2)}(\varepsilon_2|\varepsilon_1,\varepsilon_0) = t^{(1)}(\varepsilon_2|\varepsilon_1)$$

# FROM DYNAMICS TO GEOMETRY




- **SCENARIO 1: "AVALANCHES".** Big activated jump followed by sequence of smaller rearrangements. Correlations.

$$t^{(2)}(\varepsilon_2|\varepsilon_1,\varepsilon_0) > t^{(1)}(\varepsilon_2|\varepsilon_1) \Rightarrow q^{(2)}_{MAX}(\varepsilon_2|\varepsilon_1,\varepsilon_0) > q^{(1)}_{MAX}(\varepsilon_2|\varepsilon_1)$$

- **SCENARIO 2: "MEMORYLESS".** Second jump uncorrelated to the first.

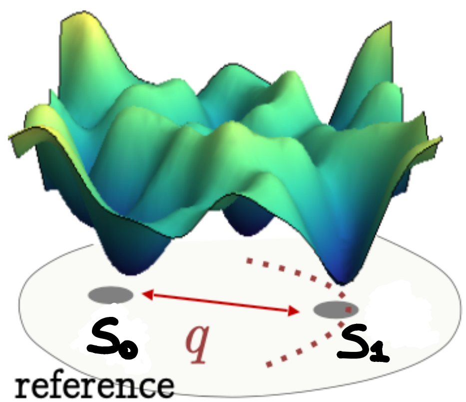
$$t^{(2)}(\varepsilon_2|\varepsilon_1,\varepsilon_0) = t^{(1)}(\varepsilon_2|\varepsilon_1) \Rightarrow q^{(2)}_{MAX}(\varepsilon_2|\varepsilon_1,\varepsilon_0) = q^{(1)}_{MAX}(\varepsilon_2|\varepsilon_1)$$

# Three-point complexity: landscape's transitions

# THE TWO-POINT COMPLEXITY

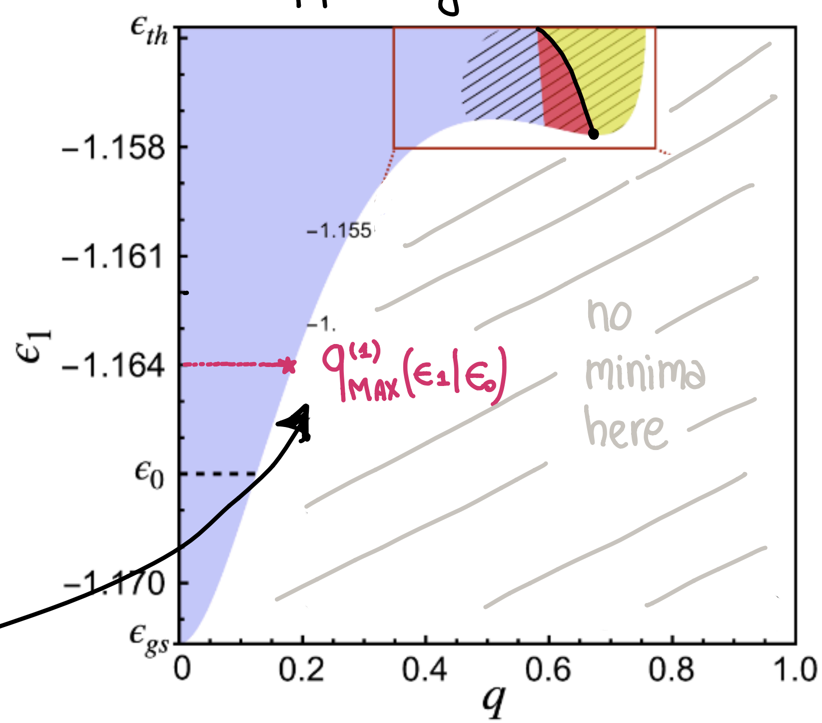
- Select a minimum $s_0$ with energy $\varepsilon_0$, with flat distribution
- count number $\mathcal{N}(\epsilon_1, q_{01}|\epsilon_0)$ of minima $s_1$ at energy $\epsilon_1$, overlap $q_{01} = \frac{s_0 \cdot s_1}{N} \equiv q$
- Two-point complexity: $\Sigma^{(2)}(\epsilon_1, q|\epsilon_0) = \lim_{N\to\infty} \frac{\langle \log \mathcal{N}(\epsilon_1, q|\epsilon_0)\rangle_0}{N}$ flat average over $s_0$

CAVAGNA, GIARDINA, PARISI 1991; VR, BIROLI, CAMMAROTA 2019

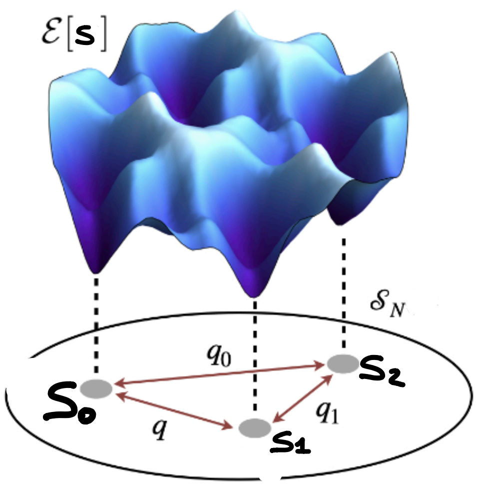



$q^{(1)}_{MAX}(\epsilon_1|\epsilon_0)$ = overlap of closest minima at energy density $\epsilon_1$

Support of $\Sigma^{(2)} > 0$



# LOCAL GEOMETRY: THREE-POINT COMPLEXITY




- Select minimum $s_0$ with energy density $\varepsilon_0$ (flat measure)
- Select minimum $s_1$ with energy $\varepsilon_1$, at overlap $q = \frac{s_0 \cdot s_1}{N}$
- Compute how many minima $s_2$ at energy $\varepsilon_2$ and overlaps $q_1, q_0$: $\mathcal{N}_{s_1,s_0} \sim e^{N\Sigma^{(3)}(\varepsilon_2, q_0, q_1 | \varepsilon_1, \varepsilon_0, q)}$

$$\boxed{\Sigma^{(3)}(\epsilon_2, q_1, q_0 | \epsilon_1, \epsilon_0, q) = \lim_{N\to\infty} \frac{1}{N} \left\langle \log \mathcal{N}_{s_0 s_1}(\epsilon_2, q_1, q_0) \right\rangle_{s_0, s_1}}$$

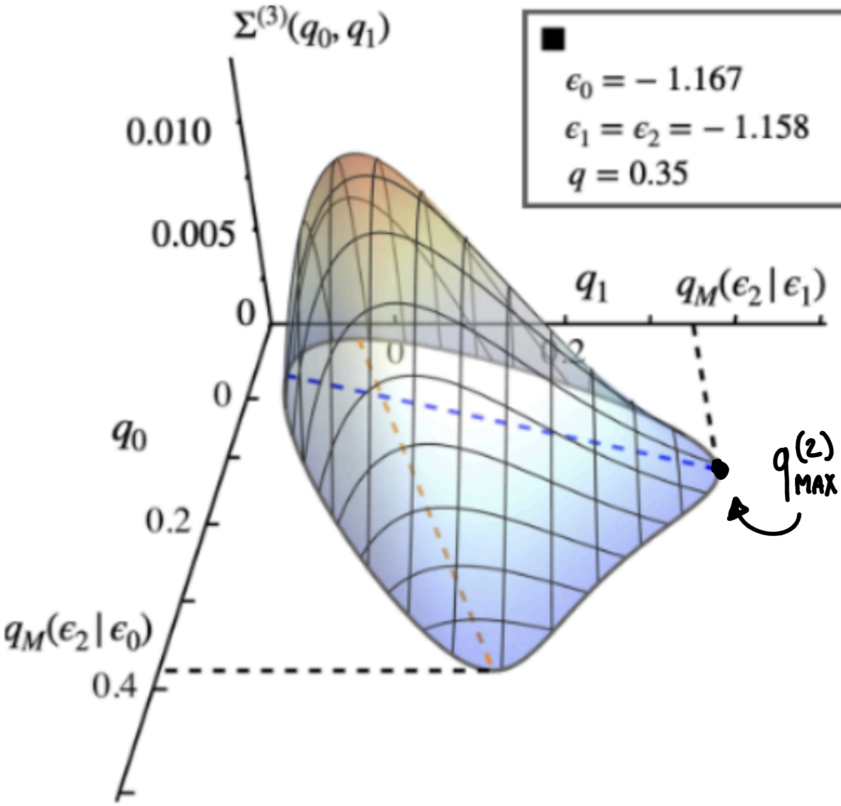
The average means:

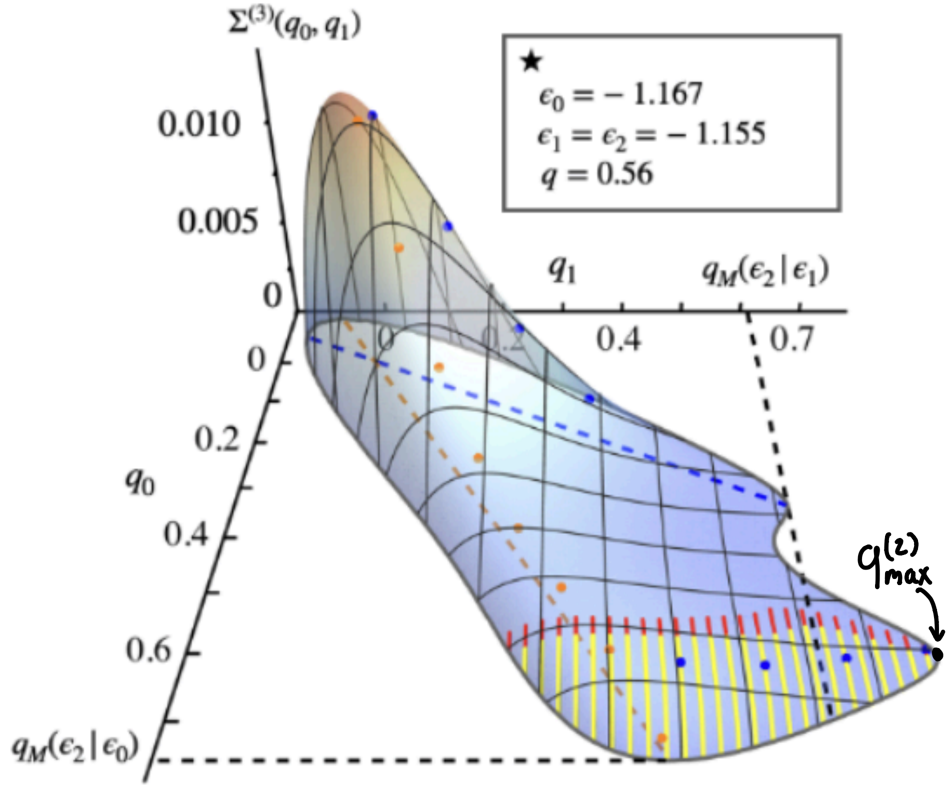
$$\langle \otimes \rangle_{s_0,s_1} = \left\langle \frac{1}{\mathcal{N}(\varepsilon_0)} \int ds_0 \, \omega_{\varepsilon_0}(s^0) \frac{1}{\mathcal{N}(\varepsilon_1|\varepsilon_0)} \int ds_1 \, \omega_{\varepsilon_1,q}(s_1|s_0) \otimes \right\rangle$$

$\omega$ - random measures selecting stationary points

The role of the minima $s_0, s_1$ is not equivalent.

# THREE-POINT COMPLEXITY



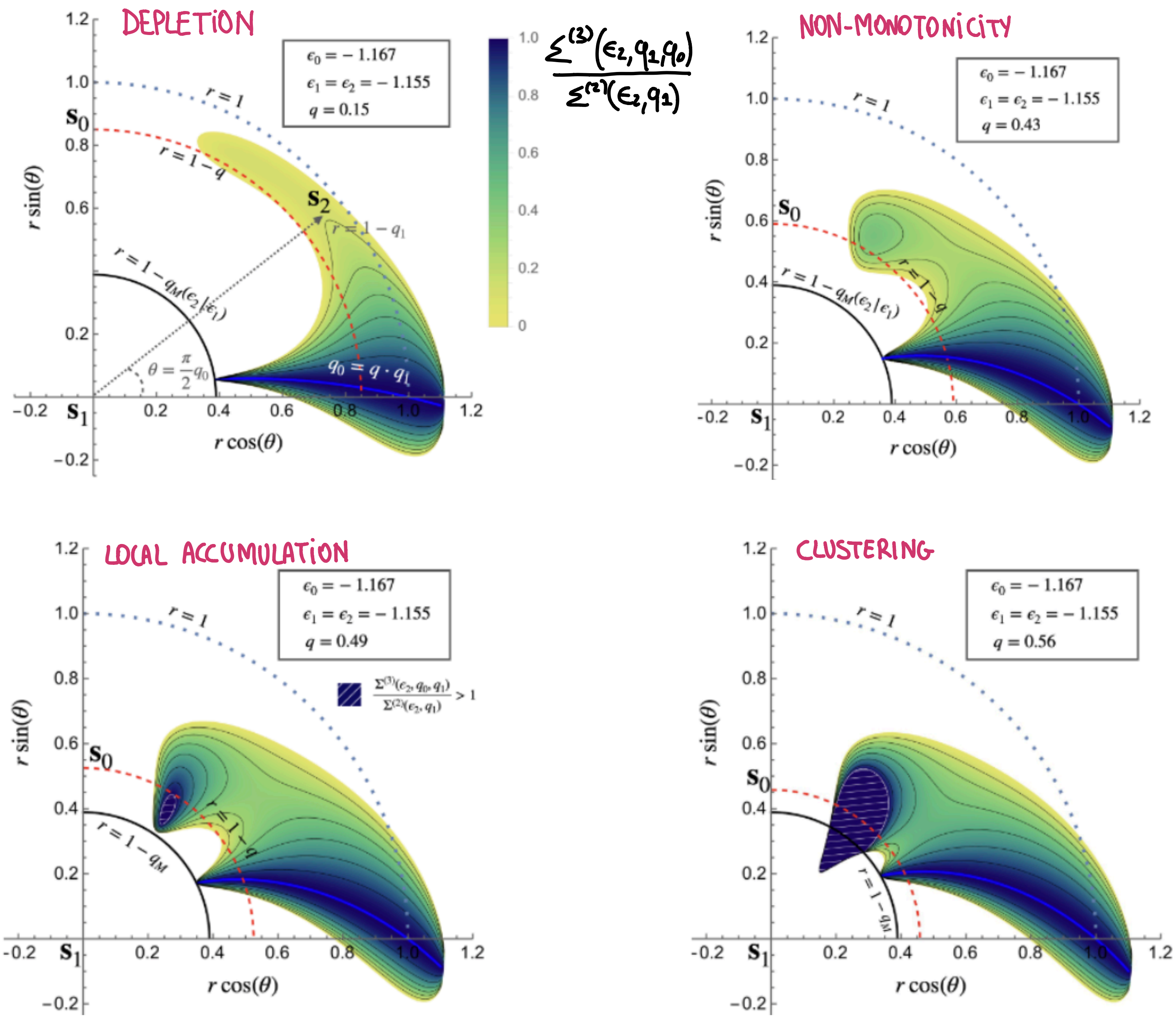




$q^{(2)}_{MAX}(\varepsilon_2|\varepsilon_1, \varepsilon_0)$ = overlap of closest stationary points at energy $\varepsilon_2$

**CLUSTERING:** $q^{(2)}_{MAX}(\varepsilon_2|\varepsilon_1, \varepsilon_0) > q^{(1)}_{MAX}(\varepsilon_2|\varepsilon_1)$ $\Longrightarrow$ scenario 1: avalanches.

**NO CLUSTERING:** $q^{(2)}_{MAX}(\varepsilon_2|\varepsilon_1, \varepsilon_0) = q^{(1)}_{MAX}(\varepsilon_2|\varepsilon_1)$ $\Longrightarrow$ scenario 2: memoryless

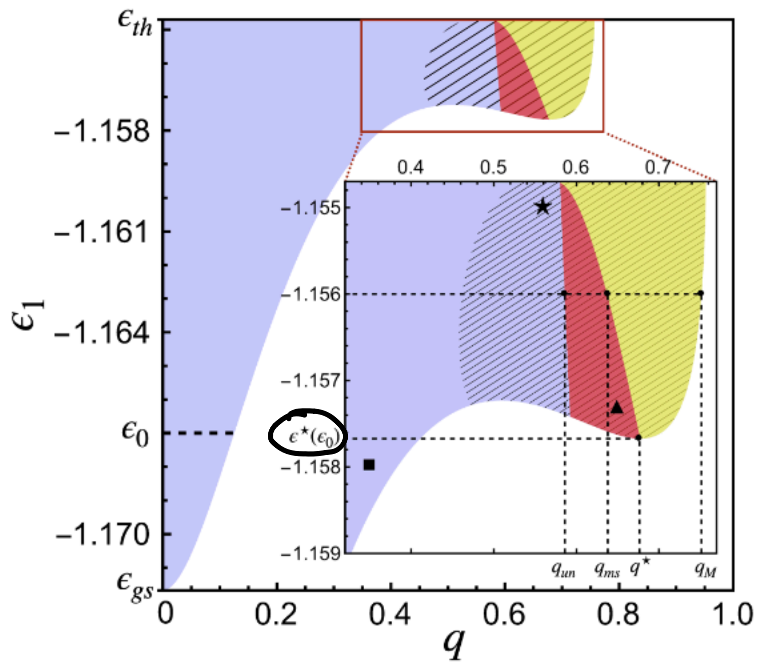
# TRANSITIONS IN LANDSCAPE

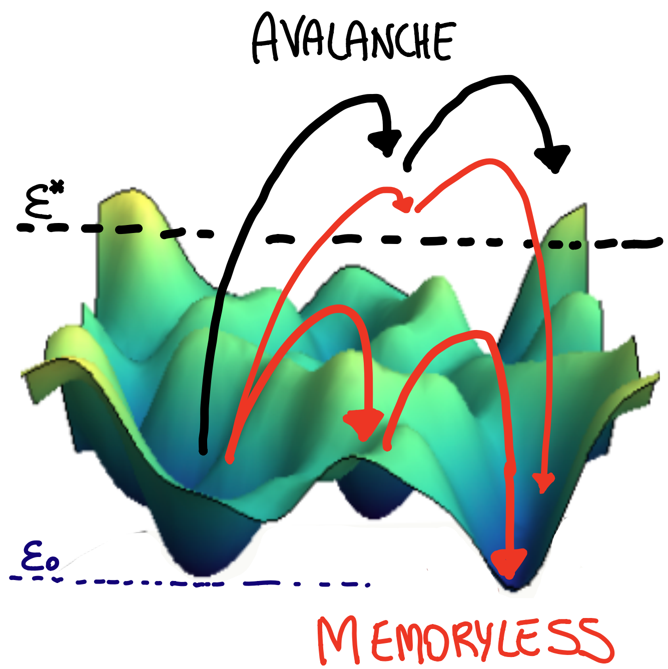


# Back to dynamics

# LANDSCAPE'S LOCAL GEOMETRY SUGGESTS THAT:

There exist a critical energy $\varepsilon^*(\varepsilon)$ $\Longleftarrow$ computed from two point complexity




Scenario 1, AVALANCHES: $\epsilon_1 > \epsilon^*(\epsilon_0)$, $\epsilon^*(\epsilon_0) < \epsilon_2 < \epsilon^*(\epsilon_1)$

Scenario 2, MEMORYLESS: all other jumps; including equal-energy paths.

THE METHOD

# Replicated Kac-Rice & Random Matrices

# REPLICATED KAC-RICE

Number stationary points: $\mathcal{N}(\epsilon_0) = \int_{S_N(\sqrt{N})} ds \, \omega_{\epsilon_0}(s)$

$$\omega_{\epsilon_0}(s) = \delta(\nabla \mathcal{E}[s]) \, \delta(\mathcal{E}[s] - N\epsilon_0) \, |\det \nabla^2 \mathcal{E}[s]|$$

review: VR, FYODOROV 2023

$$\Sigma^{(3)} = \lim_{N\to\infty} \frac{1}{N} \left\langle \frac{1}{\mathcal{N}(\epsilon_0)} \int ds_0 \, \omega_{\epsilon_0}(s^0) \, \frac{1}{\mathcal{N}(\epsilon_1|\epsilon_0)} \int ds_1 \, \omega_{\epsilon_1, q}(s_1|s_0) \, \log \mathcal{N}_{s_0 s_1}(\epsilon_2, q_1, q_0) \right\rangle$$

$\mathcal{N}$ = integrals of random measures containing coupled RM

$\boxed{1}$ replica tricks $\log x = \lim_{n\to 0} \frac{x^n - 1}{n}$, $1/x = \lim_{n\to 0} x^{n-1}$

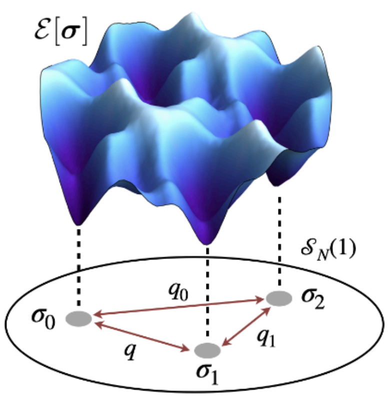
$\boxed{2}$ Take averages → Kac-Rice formulas

$\boxed{3}$ Random-matrix problem: $\left\langle \prod_a |\det \nabla^2 \mathcal{E}[s^a]| \right\rangle$

stationary points with given energies and overlaps

Large N: expectations factor!

# MINIMA OR SADDLES? RANDOM MATRICES




- The Hessian $\nabla^2\mathcal{E}(s_2)$ is a random matrix
- Functions $\beta_\alpha, \mu_\alpha$ of $q_0, q_1, q, \epsilon_0, \epsilon_1, \epsilon_2$ such that $\nabla^2\mathcal{E}(s_2)$ statistically equivalent to:

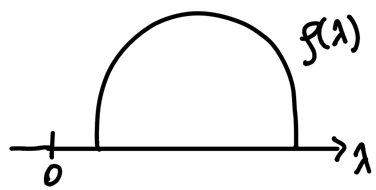
$$\nabla^2\mathcal{E}(s_2) \overset{\text{in law}}{\sim} \prod_{\alpha=N-2,N-1}\left(1-\beta_\alpha|e_\alpha\rangle\langle e_\alpha|\right) X_{GOE} \prod_{\alpha=N-2,N-1}\left(1-\beta_\alpha|e_\alpha\rangle\langle e_\alpha|\right) + \sum_{\alpha=N-2,N-1}\mu_\alpha|e_\alpha\rangle\langle e_\alpha|$$

RANK-2 MULTIPLICATIVE PERTURBATION

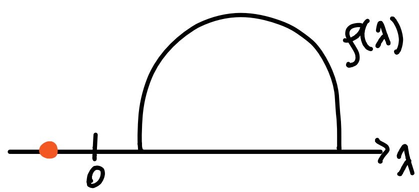
GAUSSIAN ORTHOGONAL ENSEMBLE
$\sigma^2 = p(p-1)$

RANK-2 ADDITIVE PERTURBATION

Distribution of eigenvalues:




$\Rightarrow$ Local minimum



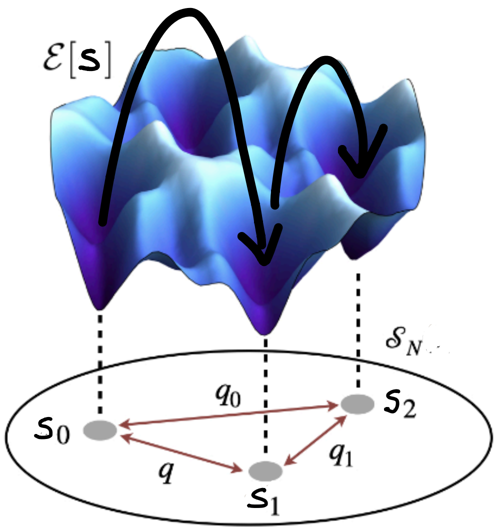

$\Rightarrow$ saddle

# SUMMARY:

- For pure spherical p-spin (baby glass): dynamics at $N \to \infty$ (mean-field) solved. Long-time, activated dynamics (non mean-field): theory challenge.

- Effective models of dynamics assume "memoryless" jumps. Landscape analysis suggests correlations matter for jumps to high-energy minima, while low-energy jumps are "memoryless".

- Quenched 3-point complexity is now known.




$\Longrightarrow$ A. Pacco, A. Rosso, V. Ros, In preparation, 2024